Chapter 21 – Electric Charge

• Historically people knew of electrostatic effects
• Hair attracted to amber rubbed on clothes
• People could generate “sparks”
• Recorded in ancient Greek history
• 600 BC Thales of Miletus notes effects
• 1600 AD - William Gilbert coins Latin term electricus from Greek ηλεκτρον (elektron) – Greek term for Amber
• 1660 Otto von Guericke – builds electrostatic generator
• 1675 Robert Boyle – show charge effects work in vacuum
• 1729 Stephen Gray – discusses insulators and conductors
• 1730 C. F. du Fay – proposes two types of charges – can cancel
• Glass rubbed with silk – glass charged with “vitreous electricity”
• Amber rubbed with fur – Amber charged with “resinous electricity”
A little more history

• 1750 Ben Franklin proposes “vitreous” and “resinous” electricity are the same ‘electricity fluid” under different “pressures”
• He labels them “positive” and “negative” electricity
• Proposes “conservation of charge”
• June 15 1752(?) Franklin flies kite and “collects” electricity
• 1839 Michael Faraday proposes “electricity” is all from two opposite types of “charges”
• We call “positive” the charge left on glass rubbed with silk
• Today we would say ‘electrons” are rubbed off the glass
Torsion Balance • Charles-Augustin de Coulomb - 1777

Used to measure force from electric charges and to measure force from gravity

\[ \tau = -\kappa \theta \] - “Hooks law” for fibers

(recall \( F = -kx \) for springs)

General Equation with damping

\[ \theta \] – angle

\[ I \] – moment of inertia

\[ C \] – damping coefficient

\[ \kappa \] – torsion constant

\[ \tau \] – driving torque

\[ I \frac{d^2 \theta}{dt^2} + C \frac{d\theta}{dt} + \kappa \theta = \tau(t) \]
Solutions to the damped torsion balance

\[ \theta = A e^{-\alpha t} \cos(\omega t + \phi) \]

General solutions are damped oscillating terms – ie damped SHO
A = amplitude
t = time
\( \alpha \) = damping frequency = 1/damping time (e folding time)
\( \phi \) = phase shift
\( \omega \) = resonant angular frequency

\[ \alpha = C/2I \]

If we assume a lightly damped system where:

\[ C \ll \sqrt{\kappa I} \]

Then the resonant frequency is just the undamped resonant frequency

\[ f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\kappa/I} \]

\( \omega_n = \sqrt{\kappa/I} \) (\( \omega_n \) = “natural undamped resonant freq”)
recall for a spring with mass \( m \) that
\( \omega = \sqrt{k/m} \) where \( k \) = spring constant
General solution with damping

- If we do NOT assume small damping then the resonant freq is shifted DOWN
- From the “natural undamped resonant freq:
  - \( \omega_n = \sqrt{\kappa/I} \)
  - Note the frequency is always shifted DOWN

\[
\omega = \sqrt{\omega_n^2 - \alpha^2} = \sqrt{\kappa/I - \left(\frac{C}{2I}\right)^2}
\]
Constant force and critical damping

- When the applied torque (force) is constant
- The drive term $\tau(t) = F \times L$ where $F$ is the force
- $L$= moment arm length

\[
\theta = \frac{F \times L}{\kappa}
\]

We want to measure the force $F$

To do this we need $\kappa$

We get $\kappa$ from measuring the resonant freq $\omega$

Then  $\kappa = \omega^2 I$

In real torsion balances the system will oscillate at resonance and we want to damp this

Critical damping (fastest damping) for  $C_c = 2\sqrt{\kappa I}$
Sinusoidal Driven Osc

\[ \frac{d^2 x}{dt^2} + 2\zeta \omega_0 \frac{dx}{dt} + \omega_0^2 x = F_0 \sin(\omega t), \]

\[ x(t) = \frac{F_0}{Z_m \omega} \sin(\omega t + \phi) \]

\[ \phi = \arctan \left( \frac{2\omega \omega_0 \zeta}{\omega^2 - \omega_0^2} \right) \]

- Max amplitude is achieved at resonance
- \( \omega_r = \omega_0 \sqrt{1 - 2\zeta^2} \)
- For a mass and spring \( \omega_0 = \sqrt{k/m} \)
- Normal damping term \( \Gamma = 2\zeta \omega_0 \)
Universal Normalized (Master) Oscillator Eq

No driving (forcing) function equation
System is normalized so undamped resonant freq $\omega_0 = 1$. $\tau = t/t_c$ $t_c =$ undamped period $\omega = \omega/\omega_0$

$$\frac{d^2q}{d\tau^2} + 2\zeta \frac{dq}{d\tau} + q = 0$$

With sinusoidal driving function

$$\frac{d^2q}{d\tau^2} + 2\zeta \frac{dq}{d\tau} + q = \cos(\omega\tau).$$

We will consider two general cases
Transient $q_t(t)$ and steady state $q_s(t)$
Transient Solution

\[
q_t(\tau) = \begin{cases} 
  e^{-\zeta \tau} \left( c_1 e^{\tau \sqrt{\zeta^2 - 1}} + c_2 e^{-\tau \sqrt{\zeta^2 - 1}} \right) & \zeta > 1 \text{ (overdamping)} \\
  e^{-\zeta \tau} (c_1 + c_2 \tau) = e^{-\tau} (c_1 + c_2 \tau) & \zeta = 1 \text{ (critical damping)} \\
  e^{-\zeta \tau} \left[ c_1 \cos \left( \sqrt{1 - \zeta^2} \tau \right) + c_2 \sin \left( \sqrt{1 - \zeta^2} \tau \right) \right] & \zeta < 1 \text{ (underdamping)}
\end{cases}
\]
Steady State Solution

\[ \frac{d^2 q}{d\tau^2} + 2\zeta \frac{dq}{d\tau} + q = \cos(\omega \tau) + i \sin(\omega \tau) = e^{i\omega \tau}. \]

\[ q_s(\tau) = Ae^{i(\omega \tau + \phi)}. \]

\[ q_s = Ae^{i(\omega \tau + \phi)}, \quad \frac{dq_s}{d\tau} = i\omega Ae^{i(\omega \tau + \phi)}, \quad \frac{d^2 q_s}{d\tau^2} = -\omega^2 Ae^{i(\omega \tau + \phi)}. \]

\[ -\omega^2 Ae^{i(\omega \tau + \phi)} + 2\zeta i\omega Ae^{i(\omega \tau + \phi)} + Ae^{i(\omega \tau + \phi)} = ( -\omega^2 A + 2\zeta i\omega A + A) e^{i(\omega \tau + \phi)} = e^{i\omega \tau}. \]

\[ -\omega^2 A + 2\zeta i\omega A + A = e^{-i\phi} = \cos \phi - i \sin \phi. \]
Steady State Continued

\[ A(1 - \omega^2) = \cos \phi \quad 2\zeta \omega A = -\sin \phi. \]

\[ A^2(1 - \omega^2)^2 = \cos^2 \phi \]
\[ (2\zeta \omega A)^2 = \sin^2 \phi \]
\[ \Rightarrow A^2[(1 - \omega^2)^2 + (2\zeta \omega)^2] = 1. \]

\[ A = A(\zeta, \omega) = \frac{1}{\sqrt{(1 - \omega^2)^2 + (2\zeta \omega)^2}}. \]
Solve for Phase

\[ \tan \phi = -\frac{2\zeta \omega}{1 - \omega^2} = \frac{2\zeta \omega}{\omega^2 - 1} \Rightarrow \phi \equiv \phi(\zeta, \omega) = \arctan \left( \frac{2\zeta \omega}{\omega^2 - 1} \right). \]

Note the phase shift is frequency dependent
At low freq \( \phi \rightarrow 0 \)
At high freq \( \phi \rightarrow 180 \) degrees
Remember \( \omega = \omega/\omega_0 \)
Full solution

\[ q_s(\tau) = A(\zeta, \omega) \cos(\omega \tau + \phi(\zeta, \omega)) = A \cos(\omega \tau + \phi). \]

\[ q(\tau) = q_t(\tau) + q_s(\tau). \]
Amplitude vs freq – Bode Plot

\[ A = A(\zeta, \omega) = \frac{1}{\sqrt{(1 - \omega^2)^2 + (2\zeta\omega)^2}}. \]

Frequency response of ideal harmonic oscillator
## Various Damped Osc Systems

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Undamped \textbf{resonant frequency} : \( f_n \)

\[
\frac{1}{2\pi} \sqrt{\frac{K}{M}} \quad \frac{1}{2\pi} \sqrt{\frac{\mu}{I}} \quad \frac{1}{2\pi} \sqrt{\frac{1}{LC}} \quad \frac{1}{2\pi} \sqrt{\frac{1}{LC}}
\]

Differential equation:

\[
M\ddot{x} + \gamma \dot{x} + Kx = F \quad I\ddot{\theta} + \Gamma \dot{\theta} + \mu \theta = \tau \quad L\ddot{q} + R\dot{q} + \frac{q}{C} = e \quad C\ddot{e} + \dot{e}/R + \frac{e}{L} = i
\]
Gold leaf electroscope – used to show presence of charge
Gold leaf for gilding is about 100 nm thick!!
Leyden Jar – historical capacitor
Force between charges as measured on the lab with a torsion balance

\[ F_C = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \]

\[ \varepsilon_0 \sim 8.854 \ 187 \ 817 \ \ldots \ \times 10^{-12} \quad \text{Vacuum permittivity} \]

\[ \varepsilon_0 = \frac{1}{\mu_0 c_0^2} \]

\[ \mu_0 = \text{Vacuum permeability (magnetic)} \]

\[ = 4\pi \times 10^{-7} \ \text{H m}^{-1} – \text{defined exactly} \]

\[ c_0 = \text{speed of light in vacuum} \]
Coulombs “Law”

Define the electric field $E = F/q$ where $F$ is the force on a charge $q$.
In the lab we measure an inverse square force law like gravity.

For a point charge $Q$ the $E$ field at a distance $r$ is given by
Coulomb’s Law. It is a radial field and points away from a positive
charge and inward towards a negative charge.

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \quad (1)$$
Similarity to Newton’s “Law” of Gravity
Both Coulomb and Newton are inverse square laws

\[ F = G \frac{Mm}{r^2} \hat{r} = mg \]

\[ F = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{r^2} \hat{r} = qE \]
Two charges – a dipole

\[ = 8.854187817... \times 10^{-12} \]
Energy density in the electric field

\[ u = \frac{1}{2} \varepsilon |E|^2, \]

Energy per unit volume J/m³

\[ \frac{1}{2} \varepsilon \int_V |E|^2 \, dV, \]

Total energy in a volume - Joules
Dipoles

Electric Field Lines - Equipotentials
Dipole moment definition

We define the dipole moment $\mathbf{p}$ (vector) for a set of charges $q_i$ at vector positions $\mathbf{r}_i$ as:

$$
\mathbf{p} = \sum_{i=1}^{N} q_i \mathbf{r}_i .
$$

For two equal and opposite charges ($q$) we have $\mathbf{p} = q \mathbf{r}$ where $\mathbf{r}$ is the distance between them.
Multipole Expansions

\[ f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_l^m Y_l^m(\theta, \phi). \]

General spherical harmonic expansion of a function on the unit sphere \( Y_{lm} \) functions are called spherical harmonics – like a Fourier transform but done on a sphere not a flat surface. \( C_{lm} \) are coefficients.

- \( l=0 \) is a monopole
- \( l=1 \) is a dipole
- \( l=2 \) is a quadrupole
- \( l=3 \) is an octopole (also spelled octupole)