Blackbody Radiation

A blackbody is a body that absorbs all radiation incident on it.

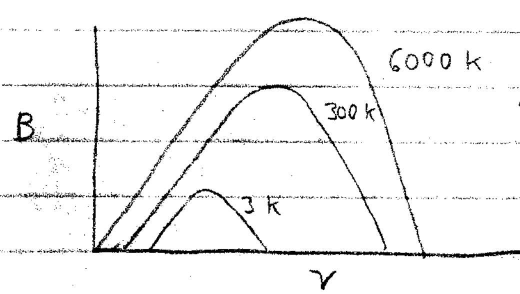
Stefan's law - For a body at temperature T the power emitted per unit area E is:

$$E = \sigma T^4 \qquad T \text{ in } {}^\circ K$$

$\sigma = 5.67 \times 10^{-8}$ w/m^2-K^4

σ - Stefan's constant

T	B	
3 K	4.6 μw/m^2	Background radiation
300 K	4.6×10^2 w/m^2	Room temp
6000 K	7.3×10^7 w/m^2	Sun



B(λ) - watts/m^2-m

Wien displacement law

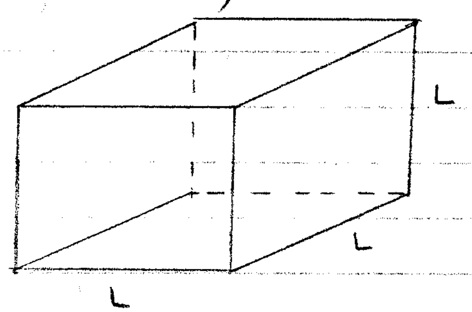
λ for max in B(λ) is inversely proportional to temperature

$$\lambda_m T = 0.29 \text{ cm-K}$$

T	λ_m
3 K	1 mm
300 K	10 μm
6000 K	5000 Å

Planck Radiation Law

Consider a metal rectangular cavity cube of side L



Recall $\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 = \nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$

Cavity solutions are as follows
(E must be 0 at walls (also B))

$$E_n = C e^{-i\omega t} \sin(n_x \pi x/L) \sin(n_y \pi y/L) \sin(n_z \pi z/L)$$

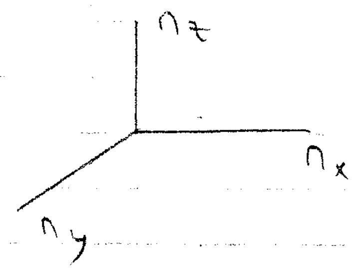
n_x, n_y, n_z – mode #'s

$$\frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2) = \frac{\omega^2}{c^2}$$

let $n^2 = n_x^2 + n_y^2 + n_z^2$

$$\frac{\pi}{L} n = \frac{\omega}{c} \qquad n = \frac{L\omega}{\pi c}$$

n_x, n_y, n_z are to be identified with photon modes



$D(n)\,dn$ – # photon modes between n & $n+dn$

Note $-n_x, -n_y, -n_z$ is the same mode as n_x, n_y, n_z so only one octant is unique (ie $1/8$)

$$D(n)\,dn = \frac{1}{8} 4\pi n^2\,dn \times 2 \text{ (polarization)}$$
$$= \pi n^2\,dn$$

$D(\omega)\,d\omega$ – # photon modes between ω & $\omega+d\omega$

$$D(\omega)\,d\omega = D(n)\,dn$$

$$D(\omega) = D(n)\,{dn}/{d\omega} \qquad n = \frac{L\omega}{\pi c}$$

$$D(\omega) = \pi\left(\frac{L}{\pi c}\right)^3 \omega^2 = \frac{V\omega^2}{\pi^2 c^3}$$

V – volume of cavity

Ascribe to each mode n
energy $n \hbar \omega$
n photons of energy $\hbar \omega$ each
$\hbar = h/2\pi \quad h = 6.62 \times 10^{-34}$ J-sec
What are the average number of photon of frequency ω at temperature T

Recall from thermodynamics that the probability of a mode of energy E_n is proportional to $e^{-E_n/kT}$
$$E_n = n \hbar \omega$$

$$\langle n \rangle = \frac{\sum_0^\infty n e^{-n\hbar\omega/kT}}{\sum_0^\infty e^{-n\hbar\omega/kT}} = \frac{\sum n x^n}{\sum x^n}$$

$$x \equiv e^{-\hbar\omega/kT}$$

$$\sum n x^n = x \frac{d}{dx} \sum x^n$$

$$\sum_0^m x^n = 1 + x + x^2 + \ldots + x^m = S$$
$$Sx = x + x^2 + \ldots + x^m + x^{m+1}$$
$$S - Sx = 1 - x^{m+1} \rightarrow S = \frac{(1-x^m)}{1-x}$$
$$x^m \rightarrow 0 \text{ as } m \rightarrow \infty$$
$$\rightarrow S = \frac{1}{1-x} \quad m \rightarrow \infty$$

$$\langle n \rangle = \frac{\sum n x^n}{\sum x^n} = \frac{x \frac{d}{dx} \sum x^n}{\sum x^n} = x \frac{d}{dx} \ln \sum x^n$$

$$= x \frac{d}{dx} \ln (1-x)^{-1} = -x \frac{d}{dx} \ln (1-x)$$

$$= \frac{x}{1-x} = \frac{1}{1/x - 1} = \underline{\underline{\frac{1}{e^{\hbar\omega/kT} - 1}}}$$

$$\langle n \rangle = \frac{1}{e^{\hbar\omega/kT} - 1}$$

Planck Distribution Function

let $\rho(\omega)\,d\omega$ = energy density due to BB rad between ω & $\omega + d\omega$

$$\rho(\omega)\,d\omega = \frac{1}{V} D(\omega) \langle n \rangle \hbar\omega \, d\omega$$

$$\rho(\omega) = \frac{1}{V} D(\omega) \langle n \rangle \hbar\omega$$

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1} \hbar\omega$$

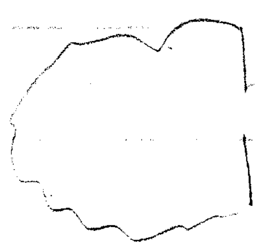
$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}$$

$$\rho(\omega)\,d\omega = \rho(\nu)\,d\nu \qquad \rho(\nu) = \rho(\omega)\, d\omega/d\nu$$

$$\omega = 2\pi\nu \qquad \rho(\nu) = 2\pi\,\rho(\omega)$$

$$\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \quad \text{joules}/m^3\text{-}Hz$$

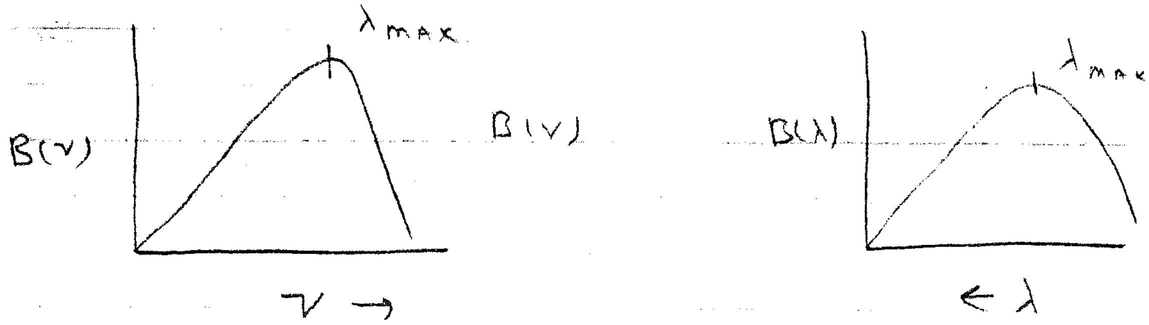
$$\hbar = h/2\pi \qquad \hbar\omega = h\nu$$



$$B(\nu) = c\,\rho(\nu)/4$$

Planck law $\longrightarrow$ $$B(\nu) = \frac{2\pi h \nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$B(\nu)$ – watts/m^2-Hz



$B(\nu)\, d\nu = B(\lambda)\, d\lambda$

$B(\nu)$ $\quad \lambda_{max} T = 0.51$ cm K

$B(\lambda)$ $\quad \lambda_{max} T = 0.29$ cm K

Raleigh Jeans $\quad h\nu \ll kT$

$$B(\nu) = \frac{2\pi h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1} \longrightarrow 2\pi kT\nu^2/c^2 = 2\pi kT/\lambda^2$$

let $S(\nu) = c\rho/4\pi \equiv$ watts/m^2-st-Hz

$S(\nu) \longrightarrow 2kT/\lambda^2$

B.B

$\vee$ A_e, Ω_e

Power detected $= S(\nu) A_e \Omega_e \beta \rightarrow \dfrac{2kT A_e \Omega_e}{\lambda^2}$

but $A_e \Omega_e = \lambda^2$

$S(\nu) \longrightarrow 2kT\beta$ $\quad$ R.J. 2 polarization

$S(\nu) \longrightarrow kT\beta$ $\quad$ one pol.

$$\rho(\nu,T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1} \quad (J/m^3\text{-}Hz)$$

this is photon energy density per freq interval

Total energy density $= \int_0^\infty \rho(\nu)\,d\nu$

$$U(T) = \int_0^\infty \rho(\nu)\,d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3\,d\nu}{e^{h\nu/kT}-1}$$

$$= \frac{8\pi k^4 T^4}{h^3 c^3}\int_0^\infty \frac{x^3\,dx}{e^x-1} \qquad x = h\nu/kT$$

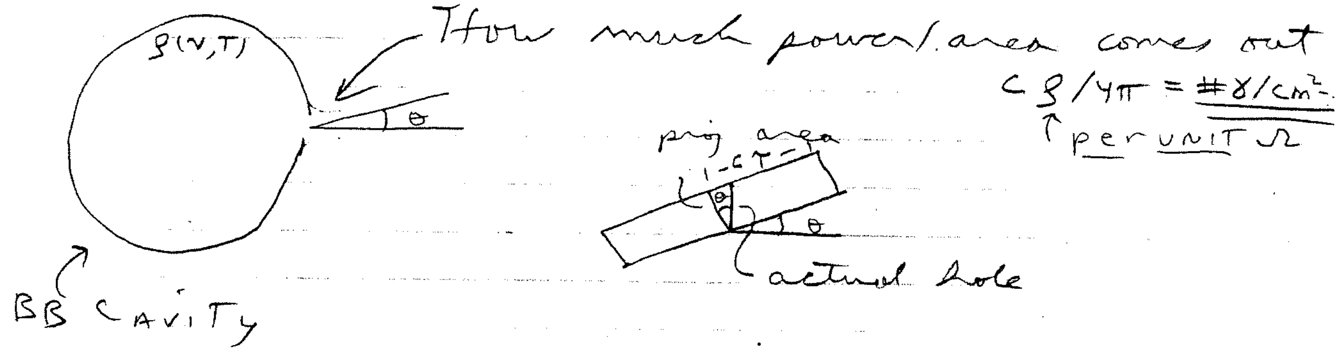
↑ NOTE $\underline{\underline{T^4}}$

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \pi^4/15$$

$$\Longrightarrow U(T) = \frac{8\pi^5 k^4}{15 h^3 c^3}\,T^4 \qquad (J/m^3)$$

This is the <u>total</u> energy density

Stephan Boltzmann Law
(Surface flux from a BB)



In time τ how much energy per ν comes out

assumes isotropic

$$E_\nu(\tau) = \underbrace{\int}_{\text{over } \theta, \phi} \rho(\nu,T)\, \underbrace{A\cos\theta}_{\text{projected area}}\, \underbrace{c\tau}_{\text{length of photon "stream"}}\, \frac{d\Omega}{4\pi}$$

$$\int \cos\theta\, d\Omega = \int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\, d\phi = 2\pi \int_0^{\pi/2} \tfrac{1}{2}\sin 2\theta\, d\theta$$

$$= -\pi \left.\frac{\cos 2\theta}{2}\right|_0^{\pi/2} = \pi$$

$$\Rightarrow \quad E_\nu(\tau) = \frac{c}{4}\rho(\nu,T)\, A\, \tau$$

NOTE $\underline{c/4}$

Flux from hole $\quad F_\nu = \dfrac{E_\nu(\tau)}{A\tau} = \dfrac{c\rho(\nu,T)}{4}$

F_ν (W/cm^2 - Hz)

Power per unit area $\quad B = \int_0^\infty F_\nu\, d\nu$ (W/cm^2)

$$= \frac{c}{4}\int_0^\infty \rho(\nu,T)\, d\nu = \frac{c\,U(T)}{4} = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4$$

$$\equiv \sigma T^4 \qquad \sigma \equiv \frac{2\pi^5 k^4}{15 h^3 c^2} \equiv \text{Stephan Boltzmann Constant}$$

$\sigma \cong 5.67 \times 10^{-12}$ W/cm^2-deg^4

$B(T = 300\text{K}) \sim 46$ mW/cm^2

Similarly for photon # density

$$n(\nu, T) = \rho(\nu, T)/h\nu \quad \text{\# density per Hz}$$

$$N(T) = \int_0^\infty n(\nu, T)\, d\nu = \text{total \# density}$$

$$n(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/kT}-1} \quad (\gamma/cm^3\text{-}Hz)$$

$$N(T) = \frac{8\pi k^3 T^3}{h^3 c^3} \int_0^\infty \frac{x^2}{e^x - 1}\, dx \quad (\gamma/cm^3)$$

Note T^3

$$\int_0^\infty \frac{x^2}{e^x-1}\, dx = 2\zeta(3) \sim 2.404$$

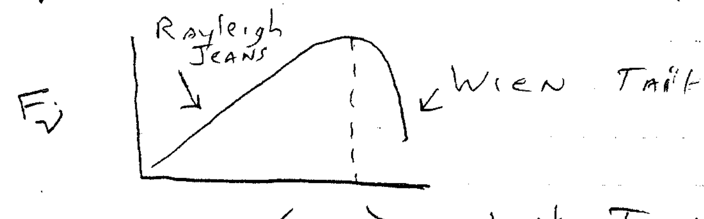
$$\Rightarrow N(T) \sim 20\, T^3 \; (\gamma/cm^3)$$

$N(T = 300) \sim 5.4 \times 10^8\ \gamma/cm^3$		
$N(T = 77) \sim 9.1 \times 10^6$	LN	
$N(T = 4.2) \sim 1.5 \times 10^3$	LHe	
$N(T = 2.7) \sim 400$	CMB	
$N(T = 1) \sim 20$	Pumped LHe^4	
$N(T = 0.1) \sim 0.02$	ADR / He^3-He^4	
$N(T = 0.005) \sim 2.5 \times 10^{-6}$	Low He^3-He^4	

Wien Displacement Law

(max of F_ν)

$$F_\nu = c\,\rho(\nu,T)/4 = (W/cm^2-Hz)$$



what is ν_{max}

$$\frac{dF_\nu}{d\nu} = 0 \Rightarrow \nu_{max}$$

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} = \frac{2\pi k^3T^3}{h^2c^2}\frac{x^3}{e^x-1}$$

$$x = h\nu/kT$$

$$\frac{dF_\nu}{d\nu} = 0 \quad \text{when} \quad \frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = 0$$

$$\frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = 0 \Rightarrow 3(1-e^{-x}) = x$$

must solve numerically

$$\Rightarrow x \sim 2.82$$

$$h\nu/kT = 2.82 \qquad \nu_{max} = \frac{2.82\,kT}{h}$$

$$\lambda_{max} = \frac{c}{\nu_{max}} = \frac{hc}{2.82\,kT} = \frac{0.51}{T}\ (cm)$$

$$\Rightarrow \lambda_{max}T \sim 0.51\ (cm-k)$$

5.1×10^7 Å$-k$

$\sim 5100\ \mu-k$

T (k)	λ_{max} For F_ν	λ_{max} For F_λ
300	17 μ	9.7 μ
77	66 μ	38 μ
4.2	1.2 mm	0.69 mm
2.7	1.9 mm	1.1 mm
1	5.1 mm	2.9 mm
0.1	51 mm	29 mm

F_ν vs F_λ

ρ_ν vs ρ_λ

Let define ρ_λ, F_λ such that

$$\rho_\nu \, d\nu \equiv -\rho_\lambda \, d\lambda \qquad F_\nu \, d\nu \equiv -F_\lambda \, d\lambda$$

$$\Rightarrow \rho_\lambda \equiv -\rho_\nu \frac{d\nu}{d\lambda} \qquad \nu = c/\lambda$$

$$d\nu/d\lambda = -c/\lambda^2 = -\nu^2/c$$

$$F_\lambda \equiv -F_\nu \frac{d\nu}{d\lambda} \qquad F_\lambda = c\rho_\lambda/4$$

$$\therefore \rho_\lambda = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1} \frac{\nu^2}{c} = \frac{8\pi h\nu^5}{c^4} \frac{1}{e^{h\nu/kT}-1}$$

$$F_\lambda = \frac{2\pi h\nu^5}{c^3} \frac{1}{e^{h\nu/kT}-1} = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT}-1}$$

└ Note different functional form

$$F_\nu \propto \frac{\nu^3}{e^{h\nu/kT}-1}$$

$$F_\lambda \propto \frac{\nu^5}{e^{h\nu/kT}-1}$$

$$\frac{dF_\nu}{d\nu} = 0 \not\Rightarrow \frac{dF_\lambda}{d\lambda} = 0$$

$$dF_\nu/d\nu = 0 \Rightarrow \lambda T \sim 0.51 \text{ cm-K}$$

$$dF_\lambda/d\lambda = 0 \Rightarrow \lambda T \sim 0.29 \text{ cm-K}$$

Not the same !!

$\hookrightarrow$ 2900 μ K

2.7×10^7 Å-K