Notes for PHYS 134: Observational Astrophysics

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May 22, 2012

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1 Astrophysical Measurements

1.1 Important Scales and Constants

Astronomers typically deal in the cgs (centimeter-gram-second) system of units, rather than the mks (meter-kilogram-second) system you may be used to. As such, lengths will be measured in centimeters (cm), and masses in grams (g). Other derived units exist, too, such as the **dyne**, which is the unit of force equal to g cm/s^2 and the **erg**, which is a unit of energy, equal to g cm^2/s^2. In the following tables, many length scales, timescales, and other fundamental units are presented that you should familiarize yourself with.

Length (cm)	Comments
10^{-33}	Planck Length
10^{-13}	Proton (nucleus) size
10^{-8}	Atomic radius
10^{-4}	"Large" molecules
10^{0}	Common experience (1 cm)
10^{3}	Largest known living things
10^{5}	Asteroid; neutron star
10^{9}	Planet
10^{11}	Star (sun)
10^{14}	Red giant
10^{15}	Solar System
10^{18}	1 light year (ly)
10^{21}	Globular cluster (bound stars)
10^{23}	Galaxies
10^{25}	Cluster of Galaxies (Virgo)
10^{28}	Size of Universe

Table 1: Relevant length scales in astrophysics.

The radius of the universe is about 10^{61} Planck lengths in width, or about 10^{41} proton widths across. The latter is more relevant to our purposes since we can typically only probe on proton length scales. Nearly all lenghscales between 10^{-33} to 10^{-13} cm are largely unexplored. We have only looked at 10^{-20} of the universe!

Constant	Symbol	Values
Reduced Planck Constant	$\hbar$	1.05×10^{-27} erg s
Gravitational Constant	G	6.67×10^{-8} cm^3/g/s^2
Speed of Light	c	3.00×10^{10} cm/s

Table 2: Fundamental units in cgs.

Finally, we come to some important time scales, shown in Table 3

Time (s)	Comments
10^{-43}	Planck Time $\left(\frac{\hbar G}{c^5}\right)$
10^{-34}	Period of highest energy cosmic ray
10^{-21}	Period of typical nuclear gamma ray
10^{-15}	Typical electron orbital period
10^{-9}	H spin flip transition photon period
10^{-3}	Audio
10^{0}	Common time perception
10^{5}	Bacteria, virus lifetimes
10^{9-10}	Large mammals
10^{13}	Largest star lifetimes
10^{17-18}	Age of universe

Table 3: Important time scales in astrophysics.

In Planck units, the age of the univers is about $10^{61}\, t_{\text{Planck}}$.

1.2 Units of Length and Angle

Due to the large distances encountered in space, astronomers will often use the unit of the **light year**, which, unsurprsingly, is the distance traveled by light in one year. If you ever run in to someone thinking that a "light year" is a unit of time, you should smack them. Hard. A light year in cm can be calculated pretty easily:

$$1\text{ ly} = c(1\text{ year}) = \left(3.00\times10^{10}\text{ cm/s}\right)\left(\pi\times10^{7}\text{ s}\right) \approx 9.46\times10^{17}\text{ cm} = 9.46\times10^{12}\text{ km} \qquad (1.1)$$

(Pro tip: the approximation of a year being approximately $\pi \times 10^7$ s is actually not too bad, and it's very easy to remember). As an order of magnitude estimate, the total distance all cars have ever driven is approximately 10 light years.

Even more common than the light year is the **parsec**. A parsec is defined as the distance from the sun you would have to be in order for the angular distance between the earth and the sun to be arsecond. This somewhat strange choice of measurement was made so that an object at a distance of 1 parsec has a parallax angle of one arsecond (hence the name "par"-"sec"). Figure 1 shows schematically how the parsec is defined.

Finally we should mention the astronomical unit (AU), which is the average distance between the earth and the sun. It is very useful when speaking of distances within the solar system. In centimeters, it is approximately 1 AU = 1.496×10^{13} cm

In observational astrophysics, we often denote the "size" of an object by its angular width in the sky. As we should all know, there are 360° in one circle. However, the degree is often too large of a unit of angle for our purposes. Recall how degrees, arcminutes, arseconds, and radians are all

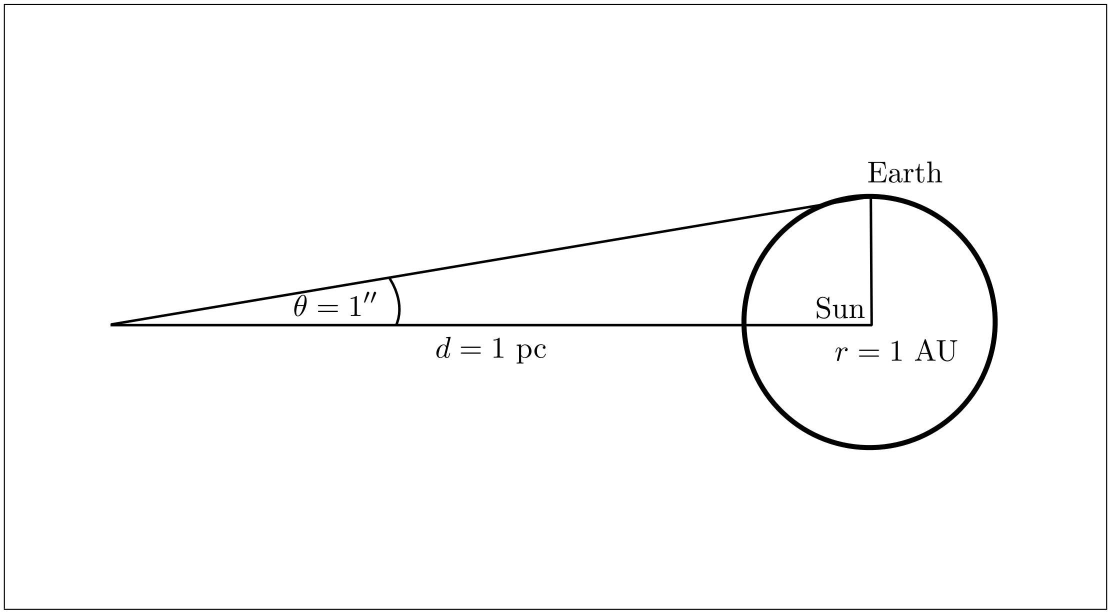

Figure 1: Schematic drawing showing how the parsec is defined.

related:

$$1^\circ = 60' \tag{1.2}$$
$$1' = 60" \tag{1.3}$$
$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ = 206,280" \tag{1.4}$$

Using some simple trigonometry, we can use the definition of a parsec to determine its length in light years. We may use the small angle approximation to say that $\tan\theta \approx \sin\theta \approx \theta$ (remember, the angle must be in radians!). Then the tangent of the angle subtended by the solar system at a distance of one parsec is

$$\sin 1" = \frac{1 \text{ AU}}{1 \text{ pc}} \quad \Rightarrow \quad 1 \text{ pc} = \frac{1 \text{ AU}}{206,280^{-1}} = 206,280\,\text{AU} \approx 3.26 \text{ ly} \tag{1.5}$$

1.3 Diffraction and Angular Resolution

All telescopes, to some extent, are just a hole through which light must pass and be collected. Passing through any hole, light is diffracted into a bessel function pattern. This will pose a fundamental limit on how resolved any image from a given telescope will be.

The degree to which a photon is diffracted depends on its wavelength. For visual perception, optical wavelengths are most used, with violet photons having a wavelength of around $\lambda = 4000$ Å =

Distance	Comments
1.3 pc	Closest Star (α Centauri)
1.3 light seconds	Earth to Moon
8.3 light minutes	Earth to Sun
5.5 light hours	Pluto
4.2 ly	Closest Star
2.5×10^4 ly	To Galactic Center
10^5 ly	Galactic Diameter
2×10^6 ly	Andromeda (M31)
10^{10} ly	Most distant observed galaxy
2×10^{10} ly	Size of Universe

Table 4: Large distance scales in astrophysics. The last two depend on distance indicators, which is a major problem in observational astronomy.

400 nm = 0.4 μm and red photons having wavelengths around $\lambda = 7000$ Å = 700 nm = 0.7 μm. The cones in your eyes respond more to color, but depend on having bright light, whereas the rods behave well in low light, but do not detect colors. Thus, galaxies and nebulae (low-light objects) typically will appear as black and white objects to human eyes, even when viewed through a telescope. Note, though, that rods sensitivity peaks more towards the blue, and less (almost to zero) towards the red. The cones, however, are somewhat reversed. Thus, objects in bright light will have inverted apparent brightness to your eyes when the intrinsic brightness is reduced. In addition, your retina is deficient in rods, so your low-light sensitivity is actually *off-center* (i.e., looking slightly away from an object makes it appear brighter). Figure 2 shows the relative responses of the rods and cones to light of various optical wavelengths.

Aside from these limitations of your eyes, they are significantly diffraction limited. For example, our eyes can resolve a planet, but stars are so small (in angular size), that we can't resolve their shape (we'll later see that even with perfect eyes, this would be quite difficult with the atmosphere). The limiting resolution of any aperture is given approximately by

$$\theta_{\text{D.L.}} \approx \frac{\lambda}{d} \tag{1.6}$$

For the case of blue light in your eye, we find

$$\theta_{\text{D.L.}} \approx \frac{0.5 \times 10^{-4}\ \text{cm}}{0.5\ \text{cm}} \sim 10^{-4}\ \text{radians} \sim 0.3' \tag{1.7}$$

So the full angle that you can resolve is $2\theta_{\text{D.L.}} \approx 0.6'$. In truth, the correct diffraction-limited angle for a circular aperture is $\theta_{\text{D.L.}} = 1.22\lambda/d$. Also worthy of note is that in low light, your iris opens more, increasing aperture, causing a higher resolution for your eyes. Thus, sunglasses can actually improve resolution (though they will saturate some of the light).

As an example of the use of this, let's investigate how small of a distance your eyes can resolve at a distance of 100 meters:

$$\sin 2\theta_{\text{D.L.}} = \frac{d}{L} = \frac{d}{10^4\ \text{cm}} \quad \Rightarrow \quad d = (10^4\text{cm}) \sin 0.6' \approx 2\ \text{cm} \tag{1.8}$$

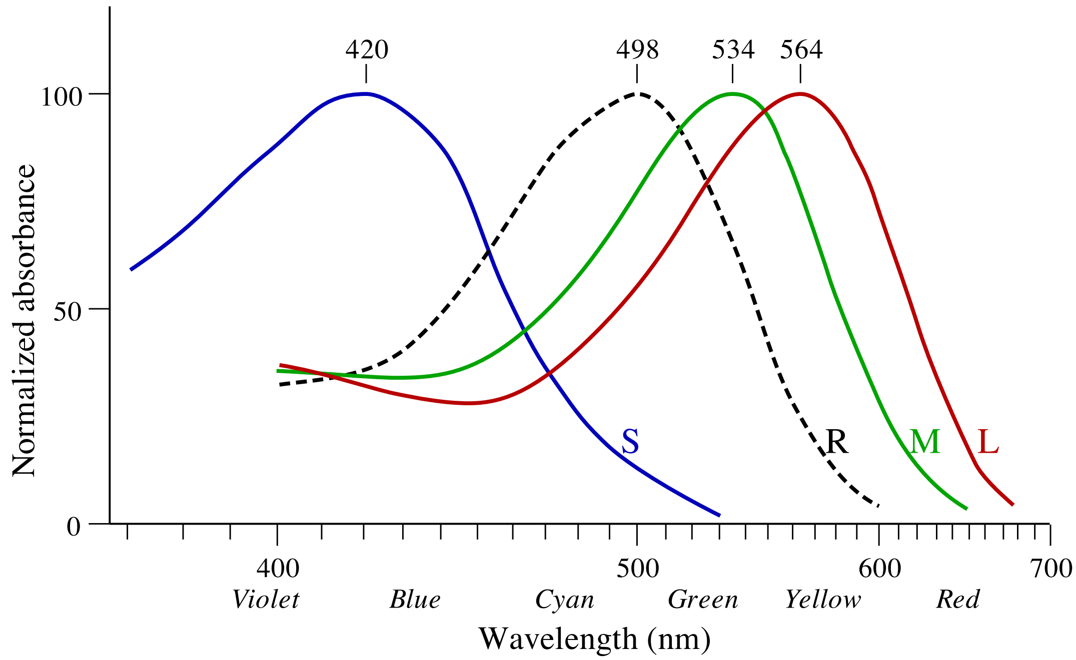

Figure 2: Response of Rods and Cones in your eye to various wavelengths of light. The dotted line is the response of the rods (colorless). The three other show the response of each of the different types of cones. The net result is that they peak closer to the red than the rods. Note how the rods detect almost no red light.

So at a distance of 100 m, your eye can theoretically resolve details on the order of 2 cm! In a more astrophysical context, the sun and moon both subtend an angle of about 30', so we can easily resolve them. A set of telescopes and their angular resolutions is shown in Table 5.

Diameter of Mirror	$1.22\lambda/d$
4 inches	2"
8 inches	1"
2.4 m (S.T.)	0.008"
5 m (Palomar)	0.004"
10 m (TMT UC-CalTech)	0.002"

Table 5: Diffraction limited angles of various telescope sizes.

However, atmospheric fluctuations limit all resloutions to around 1", regardless of the aperture size. To minimize atmospheric interference, telescopes are built on high dry mountans (like Mauna Kea in Hawaii) or in Antarctica (South Pole). Alternatively, NASA has launched the Hubble Space Telescope (around 1989-1990). By observing from space, the atmospheric effects are removed, yielding an angular resolution of less than 0.1” (aound 10 times better than previous ground-based efforts). Additionally, going to space eliminates weather and city light issues.

These considerations go past standard optical astronomy. To study radio astronomy, for example, astronomers use very long baseline interferometry (VLBI) to get data. Since the wavelengths are so long, the angular resolution is very low unless the aperture (baseline) is very high. For instance, two radio telescopes can be located on either side of earth, making the baseline be the diameter of earth. Then for a typical radio signal at $\lambda \sim 6$ cm, we get an angular resolution of

$$\theta_{\text{D.L.}} \sim \frac{\lambda}{D} \sim 5 \times 10^{-9} \text{ rad} \sim 0.001'' \tag{1.9}$$

This is about 1000 times better resolution that typical ground-based optical telescopes, making radio telescopes very useful for precision astronomy. We can also get better resolution by observing at higher frequency (like X-ray astronomy), by putting a telescope on the moon or one in orbit, etc. (Check out the RadioAstron project for a really extreme use of VLBI. Our own Carl Gwinn and Michael Johnson are working on this project!)

1.4 Magnitudes and Flux

Objects are characterized (ranked) in “brightness” by their **magnitude**. Historically, stars were ranked from 1 to 6 with 1 being the brightest and 6 being the dimmest (you can already see a problem that a higher number means a dimmer star). Instead of changing this system, modern astronomy has simply slapped a mathematical underpinning to the magnitude scale. We do so by requiring that a difference in five magnitudes corresponds to a star having a flux that is precisely 100 times greater. Mathematically, we compare the magnitudes to the fluxes thusly:

$$\frac{b_1}{b_2} = 100^{(m_2-m_1)/5} = \left(10^2\right)^{(m_2-m_1)/5} = 10^{\frac{2}{5}(m_2-m_1)} = 10^{0.4(m_2-m_1)} \tag{1.10}$$

Here b_1 is the flux of object 1 and b_2 is the flux of object 2, where as m_1 and m_2 are their magnitudes. Note that we have only defined magnitudes as a *relative* scale. We must pick a zero point for which to base it on. Also note that since a difference in 5 magnitudes necessitates a flux ratio of 100, a difference in magnitude of 2.5 must require a flux ratio of 10.

As a more concrete example, suppose we are given two stars with known magnitudes of $m_1 = 14.2$ and $m_2 = 23.7$ and we are asked to compute the ratio of their fluxes. We may jump immediately to (1.10):

$$\frac{b_1}{b_2} = 10^{0.4(23.7-14.2)} = 10^{0.4(9.5)} \approx 6.3 \times 10^3 \tag{1.11}$$

So object 1 is about 6.3×10^3 times brighter than object 2 (on a linear, flux-based scale, at least).

So far we’ve been careful to be vague about what we mean by flux. Depending on the situation, we might mean flux in terms of photons/cm^2/s or ergs/cm^2/s or any other sensible choice of units. While using actual units of energy per unit area per unit time is more physically motivated, the photon count scheme is often more practical since telescopes essentially count photons rather than energy (although look into MKIDs to find out how Ben Mazin’s lab is working on energy sensitive detectors).

Since the choice of magnitude scale is arbitrary up to a choice in zero point, there can also be negative magnitude stars (brighter than your zero point star, then). For instance, the sun is a magnitude −27 star and the moon is at −12 magnitude. Performing a calculation on these magnitudes similar to the one done above, we find the ratio in the fluxes between the moon and the sun to be 10^6! The fluxes for these two objects are $b_\odot = 10^3$ W/m^2 and $b_{\text{moon}} \sim 10^{-3}$ W/m^2. (Note that once we declared what magnitude the sun and moon are at, we have implicitly chosen a particular magnitude system).

Telescopes typically measure flux, so the astronomer is more interested in converting fluxes to magnitudes rather than the other way around. Inverting (1.10) gives us

$$\frac{b_1}{b_2} = 10^{0.4(m_2-m_1)} \tag{1.12}$$

$$\log\frac{b_1}{b_2} = 0.4(m_2-m_1) \tag{1.13}$$

$$m_2 - m_1 = 2.5\log\frac{b_1}{b_2} \tag{1.14}$$

Where, unless otherwise stated, it is always assumed that $\log = \log_{10}$.

The magnitude scale so far has described what is called the **apparent magnitude**, which measures how much light we receive here at earth. This does not tells us the intrinsic brightness of the object (related to the luminosity). The varying distances between earth and objects cause the apparent magnitude to vary significantly from its **absolute magnitude**, which is defined to be the apparent magnitude that would be measured if the object were located at 10 pc from earth. To differentiate between these two magnitudes, we use a lower case m to denote apparent magnitude, and a capital M to denote absolute magnitude. You could compute a magnitude (apparent or absolute) for any object, be it a star, galaxy, beachball, or a flashlight.

We've already mentioned that the brightness of an object decreases with increasing distance. This is due to the **inverse square law**. That is, flux scales as

$$F \propto \frac{1}{r^2} \tag{1.15}$$

With this in mind, we can directly relate apparent and absolute magnitude. Suppose m and b are the apparent magnitude and observed flux of the object at its true distance from earth, but M and B are the absolute magnitude and observed flux if the object were moved to 10 pc from the earth. We can just treat the "10 pc star" as another star and use our old formula to find the relationship between m and M:

$$m_2 - m_1 = 2.5\log\frac{b_1}{b_2} \quad \Rightarrow \quad M - m = 2.5\log\frac{b}{B} \tag{1.16}$$

However, we know the how the ratio of fluxes varies with distance, the inverse square law. Plugging this in to (1.16) gives us

$$M - m = 2.5\log\left(\frac{10\text{ pc}}{d}\right)^2 = 5\log\frac{10\text{ pc}}{d} = 5\log 10 - 5\log d = 5 - 5\log d \tag{1.17}$$

Where the last two forms of (1.17) can *only* be used if the distance d is in parsecs. Often you will see the difference between the apparent and absolute magnitudes denoted via

$$\mu \equiv m - M. \tag{1.18}$$

This quantity is called the **distance modulus** because it uniquely defines the distance to an object, though in and of itself, it tells you nothing about the luminosity of the object. The distance modulus comes in handy especially when dealing with objects of known absolute magnitude (so-called **standard candles**, like Type Ia supernovae). We measure an apparent magnitude and from that deduce a distance modulus, and thus a distance from the measurement.

As an example, suppose a galaxy at 10 megaparsecs (Mpc) has an apparent magnitude of 17. What is its absolute magnitude?

First we write out the distance in parsecs to make this computationally simple:

$$d = 10 \text{ Mpc} = 10^7 \text{ pc} \tag{1.19}$$

Now we just have a straightforward application of (1.17):

$$M = m + 5 - 5 \log d = 17 + 5 - 5 \log 10^7 = 22 - 35 = -13 \tag{1.20}$$

Note, though, that a galaxy is about 10^4 pc in size, so at 10 pc, it is not small. We treat it as though all of its light were from a small point source in making these calculations.

1.5 Photons

From quantum mechanics, we know that radiation energy is quantized into units called photons. At a frequency ν or wavelength λ, each photon has energy

$$E = h\nu = hc/\lambda \tag{1.21}$$

where we've used the fact that

$$c = \nu\lambda \tag{1.22}$$

for radiation. Here h is the Planck constant ($h \approx 6.63 \times 10^{-27}$ erg s). If we want to get a fast relation between the wavelength of a photon in angstroms and its energy, we get

$$E = \frac{1.99 \text{ Å}}{\lambda} \times 10^{-8} \text{ erg} \tag{1.23}$$

or in electron volts,

$$E = \frac{1.24 \text{ Å}}{\lambda} \times 10^{4} \text{ eV} \tag{1.24}$$

(An electron volt, or eV, is the amount of energy gained by an electron in passsing through a potential of one volt and has a value of 1 eV $= 1.602 \times 10^{-12}$ erg.) For example, your eye has a peak response at a wavelength of $\lambda = 5500$ Å, which corresponds to an energy of

$$E = \frac{1.99}{5500} \times 10^{-8} \text{ erg} = 3.62 \times 10^{-12} \text{ erg} \tag{1.25}$$

or, again in electron volts,

$$E = \frac{1.24}{5500} \times 10^4 \text{ eV} = 2.26 \text{ eV} \tag{1.26}$$

So far, our discussion of the magnitude system has been restricted to bolometric magnitudes. That is, the magnitude that corresponds to the total flux (integrated over all wavelengths) emanating from the object in question. If we define a bolometric magnitude at a particular flux, we have effectively set the entire magnitude scale. We define for a $m = 0$ star, the specific flux to be

$$F_\lambda(m = 0) = 3.7 \times 10^{-9} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Å}^{-1} \tag{1.27}$$

at the top of the atmosphere at $\lambda = 5,500$ Å. Note how this flux is defined as a "per wavelength" flux. That is, to get the total flux incident between two wavelengths, you'd have to perform an integral:

$$F_{12} = \int_{\lambda_1}^{\lambda_2} F_\lambda \, d\lambda. \tag{1.28}$$

To get "color" information on objects, astronomers use various filter systems. These filters only allow light to pass through a specified narrow band of wavelengths. A classic system is the Johnson system of UBV (U='Ultraviolet', B='Blue', V='Visible') filters. The V-band filter has a bandwidth close to that of your eye, at $4,000$ Å $- 7,000$ Å. Note that the center of this range is right at the magic number for a $m = 0$ star, 5500 Å. Then the total flux passing through a V filter due to a $m = 0$ star would need to be

$$F_V(m = 0) = F_\lambda(m = 0)\Delta\lambda = 3.7 \times 10^{-9} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Å}^{-1} \left(3,000 \text{ Å}\right) \approx 1 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \tag{1.29}$$

If we assume that the average photon passing through the filter indeed has a wavelength of $\lambda_{\text{avg}} = 5,500$ Å, then we may determine the photon flux (number of photons passing through a unit of area per unit time). Each photon has an energy of $E = hc/\lambda \approx 3.6 \times 10^{-12}$ erg, so then we may convert energy units to photons directly:

$$F(m = 0) = 1 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \times \frac{1 \text{ photon}}{3.6 \times 10^{-12} \text{ erg}} \approx 3 \times 10^6 \text{ photons cm}^{-2} \text{ s}^{-1} \tag{1.30}$$

1.6 Eyes and Telescopes

In ideal conditions, the eye has a maximum diameter of $0.5 - 0.7$ cm and thus an area of about 0.4 cm^2. Comapre this to the telescope at Palomar, which has a diameter of 5 meters and thus an area of about 2×10^5 cm^2, which is about 400,000 times bigger than your eye! Table 6 compares the relative detection capabilities of the human eye versus that of Mount Palomar. The takeaway here is that telescopes vastly outperform the eye in terms of photon collection, and thus detection of faint objects.

2 Signal to Noise

Of great importance in Astronomy is the **Signal to Noise Ratio** or SNR, for short. This is the raio of incident flux that is due to the object being observed and the random flux from other sources

Magnitude	Flux (photons cm^{-2} s^{-1})	Eeye (photons/s)	Palomar (photons/s)
0	3×10^6	10^6	6×10^{11}
5	3×10^4	10^4	6×10^9
10	300	100	6×10^7
15	3	1	6×10^5
20	0.03	10^{-2}	6×10^3
25	3×10^{-4}	10^{-4}	60
30	3×10^{-6}	10^{-6}	0.6

Table 6: Photon detection for the human eye and for the telescope at Mount Palomar. The eye's absolute detection limit is at around 8th magnitude, whereas Palomar's limit is around 25th magnitude.

that acts to corrupt the image. For obvious reasons, it is desirable to maximize the SNR. Since this discussion is pertinent to images taken with CCDs (Charged Coupled Devices), where photons are converted to electrons, signals are typically measured in electrons. See Table 7 for the definitions of some relevant variables.

Symbol	Quantity (units)
N_R	Readout Noise (e^-)
i_{DC}	Dark Current (e^-/s)
Q_e	Quantum efficiency (dimensionless)
F	Point Source Signal Flux on Telescope (photon s^{-1} cm^{-2})
F_β	Background Flux from Sky (photons s^{-1} cm^{-2} arcsec^{-2})
Ω	Pixel Size (arcsec) (assuming greater than seeing)
ϵ	Telescope Efficiency (dimensionless)
τ	Integration Time (s)
A	Telescope Area (cm^2)

Table 7: Variables relevant to SNR.

Using these variables, the signal can be decuced to be

$$S = F\tau A\epsilon Q_e \tag{2.1}$$

Physically, we are starting with the total integrated energy deposition per unit area (flux integrated over integration time), then we find the total energy deposited by multiplying this energy per area by the area of the telescope. However, not all the photons will make it through the telescope, so this total energy deposition is attenuated by a factor of ϵ, the telescope efficiency. Finally, not every photon is converted to an electron, so this number is attenuated by the quantum efficiency of the chip (or your eye, for that matter), and is thus multiplied by Q_e.

2.1 Sources of Noise

The calculation of the source signal is relatively straightforward (assuming you have all of the relevant information on the object being observed and your observing setup). However, the task of calculating the noise is a different matter.

There are three main sources of noise that we will consider here: dark current, readout noise, and background noise. Two of these, dark current and background noise, increase with integration time, whereas readout noise is independent of the exposure (integration) time.

All of these sources of noise are assumed to be uncorrelated (one does not affect the other), and since they are Poisson distributed, the standard deviation (which will end up being the "noise") of each quantity is equal to the square root of the quantity. That is,

$$\sigma_i = N_i = \sqrt{S_i} \tag{2.2}$$

2.1.1 Dark Current

CCDs work by having valence electrons be excited by incident photons into the conduction band and then being trapped there. This process happens in every pixel, and so at the end of the exposure, the electrons are "read out" onto a computer, say, and counted. However, photons are not the only source of excitation in these devices. Thermal fluctuations can also bump electrons into the conduction band. This is obviously a temperature-sensitive phenomenon, so most good telescopes use advanced cooling systems to cut down on dark current.

The rate at which these excitations occur *is* the dark current, i_{DC}. Thus, the signal generated by dark current is given by

$$S_{DC} = i_{DC}\tau \tag{2.3}$$

This signal is stochastic, so it is distributed randomly around the image. We cannot, then, completely correct for it. However, astronomers can somewhat mitigate this issue by taking a **dark frame** image. A dark frame is an image that is taken with the same exposure time as the "real" image, but with the shutter closed. In this way, the dark frame can be subtracted from the "real" image to remove a large chunk of the dark current noise. Obviously the dark frame doesn't completely recreate the noise, since the distribution is random, but it is better than nothing. We are left with only the fluctuations in the average dark current signal, which is the stanadrd deviation of the signal.

2.1.2 Background Noise

In between objects, the sky is not completely dark. City lights, the moon, starlight, and other sources of light pollution are scattered in the atmosphere and eventually create a diffuse background of light in the sky. This light is also captured by the telescope, but it is not wanted. Given the variables mentioned in Table 7, we can calculate the background signal to be about

$$S_\beta = F_\beta A\epsilon Q_e\Omega. \tag{2.4}$$

The reasoning behind this equation is almost exactly the same as for (2.1), just with the addition of Ω to make $F_\beta\Omega$ be the effective flux. Again, astronomers can subtract off the average background signal, but still be left with some noise due to the random distribution of the noise.

2.1.3 Readout Noise

When CCDs read out their images, not all of the electrons can be effectively removed. Some "electron sludge" is left over and is then recorded in the next image. These orphaned electrons are treated in the following images as if they were bona fide detections, causing some readout noise. There are also other random effects (themal excitations in between images, for example). This can be partially corrected by subtracting off a **bias frame**. This is an image that is taken with effectively zero exposure time, the electron sludge and other effects can be removed. This, like the dark frame, is subtracted off of the "real image", leaving only the variation in readout noise, N_R. Note that bias frames are often used to correct dark frames to make them true measurements of the thermal noise.

2.2 Computing the SNR

We can divide the sources of noise into the time-dependent signals, S_{time} and the time-independent readout noise, N_R. The time-dependent unwanted signals directly add to give

$$S_{\text{time}} = S + S_{DC} + S_\beta \tag{2.5}$$

The uncertainties in these signal sources are just the square roots of the signals themselves, giving

$$N_S = \sqrt{S} \qquad N_{DC} = \sqrt{S_{DC}} \qquad N_\beta = \sqrt{S_\beta} \tag{2.6}$$

Note that we've included a standard deviation in the source's output, since it is also stochastic. The combined action of these three noise sources can be determined by adding them in quadrature (since they are uncorrelated):

$$N_{\text{time}} = \sqrt{N_S^2 + N_{DC}^2 + N_\beta^2} = \sqrt{S + S_{DC} + S_\beta} = \sqrt{F\tau A\epsilon Q_e + i_{DC}\tau + F_\beta A\epsilon Q_e \Omega\tau} \tag{2.7}$$

For the total noise, we must also consider the readout noise, which is independent of time, so we add this to the time-dependent noise in quadrature:

$$N_{\text{tot}} = \sqrt{N_R^2 + N_{\text{time}}^2} = \left(N_R^2 + \tau(i_{DC} + F_\beta A\epsilon Q_e \Omega)\right)^{1/2} \tag{2.8}$$

Now let us denote

$$A_\epsilon = A\epsilon Q_e \tag{2.9}$$

as the "effective area" and

$$N_T = FA_\epsilon + i_{DC} + F_\beta A_\epsilon \Omega \tag{2.10}$$

as the time-dependent noise per unit time. Now we can express the signal-to-noise ratio somewhat compactly as

$$\frac{S}{N} = \frac{FA_\epsilon\sqrt{\tau}}{\left[\frac{N_R^2}{\tau} + FA_\epsilon + i_{DC} + F_\beta A_\epsilon \Omega\right]^{1/2}} = \frac{FA_\epsilon\sqrt{\tau}}{\left[\frac{N_R^2}{\tau} + N_T\right]^{1/2}} = \frac{FA_\epsilon \tau}{\left[N_R^2 + \tau N_T\right]^{1/2}}. \tag{2.11}$$

From this expression, we can see that there are two distinct regimes for noise domination. For short exposures (small τ), the noise is dominated by the readout noise, but as the exposure time is increased, the time-dependent noise factor begins to dominate. The transition time between the two regimes occurs at

$$\tau_c = \frac{N_R^2}{FA_\epsilon + i_{DC} + F_\beta A_\epsilon \Omega} = \frac{N_R^2}{N_T} \tag{2.12}$$

as you might expect. At this critical exposure time, the SNR is

$$S/N(\tau = \tau_c) = \frac{FA_\epsilon N_R}{\sqrt{2}(FA_\epsilon + i_{DC} + F_\beta A_\epsilon \Omega)} = \frac{FA_\epsilon N_R}{\sqrt{2} N_T} \tag{2.13}$$

We can also use this expression to find the time required to measure a desired S/N (note that as time goes up, S/N must increase due to the $\sqrt{\tau}$ dependence in S/N).

$$S_N \equiv S/N = FA_\epsilon \tau / \left[N_R^2 + \tau N_T\right]^{1/2} \tag{2.14}$$

$$S_N^2 (N_R^2 + \tau N_T) = F^2 A_\epsilon^2 \tau^2 \tag{2.15}$$

$$0 = F^2 A_\epsilon^2 \tau^2 - S_N^2 N_T \tau - S_N^2 N_r^2 \tag{2.16}$$

$$\tau = \frac{S_N^2 N_T \pm \sqrt{S_N^4 N_T^2 + rF^2 A_\epsilon^2 S_N^2 N_R^2}}{2F^2 A_\epsilon^2} \tag{2.17}$$

$$= \frac{S_N^2 N_T}{2F^2 A_\epsilon^w} \left[1 + \sqrt{1 + \frac{4F^2 A_\epsilon^2 N_R^2}{S_N^2 N_T^2}}\right] \tag{2.18}$$

2.3 Examples and Applications

Suppose that you are observing a $m = 20$ object in an atmosphere with 2" seeing with a CCD with the following specs:

$$\begin{aligned}
N_R &= 12 \\
i_{DC} &= 1\ e^-\ \text{s}^{-1}\ \text{pixel}^{-1} \quad \text{at } 35^\circ\ \text{C} \\
Q_e &= 0.3 \\
A &= 10^3 \\
\epsilon &= 0.5 \\
F_\beta &= 10^{-2}\ \text{photons s}^{-1}\ \text{cm}^{-2}\ \text{arcsec}^{-2} \quad \text{(ideal sky)} \\
\Omega &= 4\ \text{arcsec}^2 \\
F &= 0.03\ \text{photons s}^{-1}\ \text{cm}^{-2} \quad \text{(20th magnitude)}
\end{aligned}$$

Assuming all of this, the SNR would be calculated according to (2.11) to be, as a function of the integration time,

$$S/N = 4.5\sqrt{\tau} / \left[\frac{144}{\tau} + 4.5 + 1 + 6\right]^{1/2} \tag{2.19}$$

Now, the sky is rarely ideal, so if we assume that $F_\beta = 0.1$ (i.e., ten times the ideal sky background flux), we get instead

$$S/N = 4.5\sqrt{\tau}/\left[\frac{144}{\tau} + 4.5 + 1 + 60\right]^{1/2} \tag{2.20}$$

Table 8 shows some SNRs for various integration times in these two settings.

Integration Time (sec)	SNR ($F_\beta = 10^{-2}$)	SNR ($F_\beta = 0.1$)
1	0.4	0.3
10	2.8	1.6
100	13	5.5
1000	42	18

Table 8: SNRs for the setup described at different background fluxes.

If instead, we had a CCD with $Q_e = 0.5$ and $N_R = 10$ (all other things the same), we should find

$$S/N = 7.5\sqrt{\tau}/\left[\frac{100}{\tau} + 7.5 + 1 + 10\right]^{1/2} \qquad F_\beta = 10^{-2} \tag{2.21}$$

$$S/N = 7.5\sqrt{\tau}/\left[\frac{100}{\tau} + 7.5 + 1 + 100\right]^{1/2} \qquad F_\beta = 0.1 \tag{2.22}$$

The corresponding SNRs are found in Table 9.

Integration Time (sec)	SNR ($F_\beta = 10^{-2}$)	SNR ($F_\beta = 0.1$)
1	0.7	0.5
10	4.4	2.1
100	17	6.9

Table 9: SNRs for the same setup with a better CCD.

Note, though, that simply cranking up the integration time is not always an option. A CCD is limited in how many electrons it can store in each pixel during a given exposure before bleeding and ghosting effects start to affect the image quality. Astronomers can get around this by "stacking" multiple exposures on top of each other, effectively increasing the exposure time.

3 Photometry

3.1 Aperture Photometry

In aperture astronomy, concentric circular apertures are used to compute the sky-subtracted flux of a star. The inner circle is made large enough to cover almost all of the flux from the star and the outer one is large enough to obtain a good sky value but not too large. We assume the image to be analyzed is already flat fielded, though for some applications, this is not critical.

In general, we want to sum up the contributions of all the pixels where significant light from the star occurs. Since there are other sources of signal, such as CCD dark current, atmospheric emission, etc., we must subtract these so that the result we get is only due to the star. We call this corrected value the **sky subtracted value**.

Heuristically, we let $g(\lambda)$ represent the pixel value in A/D units from all sources (star, background, dark current, etc.), and $g_b(\lambda)$ represent the pixel value in A/D units of that same image if no star were present. This is the background value and is assumed to have the same integration time. We have also written down these quantities in a way that suggests their implicit wavelength dependence. Thus, the values will change when the filter is changed.

Thus, the sky subtracted signal (that of the star only) is

$$f(\lambda) = \sum_{\text{pixels}} [g(\lambda) - g_b(\lambda)] \tag{3.1}$$

Note that the sum is over all pixels where the star is present. We can obtain $g_b(\lambda)$ by either taking a separate exposure with no star present, but of equal time, or, as is more common, by using pixels near to where the star is located to calculate the background level.

To analyze the problem in detail, we introduce the following notation: r_i is the inner aperture radius, r_o is the outer aperture radius, IA is the inner aperture, and OA is the outer aperture. We assume that r_i and r_o are measured from the centroided star position.

Additionally, Table 10 gives some other notation that will be used. With this notation, the sky-

Symbol	Meaning
N_{IA}	Number of pixels in inner aperture
N_{OA}	Number of pixels in outer aperture
$G(j,k)$	Pre-flat-fielded image array
R	A/D counts per e^-
$N(j,k)$	$G(j,k)/R$, the pixel value in e^-

Table 10: More notation in aperture photometry.

subtracted stellar flux n in electrons is then

$$n = \frac{1}{R} \sum_{\text{pixels}} [g(\lambda) - g_b(\lambda)] = \frac{1}{R} f(\lambda) \tag{3.2}$$

In terms of the actual image arrays, we have

$$n = \sum_{IA} N(j,k) - \frac{N_{IA}}{N_{OA}} \sum_{OA} N(j,k) \tag{3.3}$$

Where the sums are now over the pixels in the inner and outer apertures, respectively.

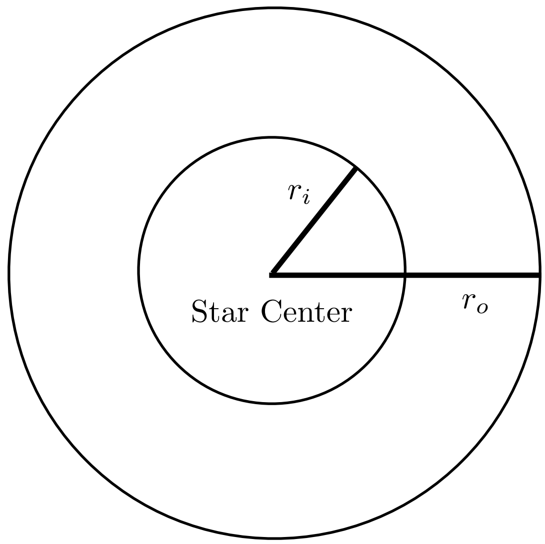

Figure 3: Notation for aperture photometry. The inner circle is the inner aperture (IA) and the outer circle is the outer aperture (OA).

3.2 Error Analysis

We can compute the error in the sky-subtracted flux using some of our notation from Section 2.2 to get

$$\delta n = \left\langle \delta n^2 \right\rangle^{1/2} = \left[\sum_{IA} (\delta N)^2 + \left(\frac{N_{IA}}{N_{OA}} \right)^2 \sum_{OA} (\delta N)^2 \right]^{1/2} \tag{3.4}$$

where

$$\delta N = \left(N_R^2 + \left(\sqrt{N} \right)^2 \right)^{1/2} = \left(N_R^2 + N \right)^{1/2} \tag{3.5}$$

Here N_R is still the readout noise and now N is the total number of electrons produced in a pixel, so δN is the uncertainty in a particular pixel's measurement. Then (3.4) becomes

$$\delta n = \left[\sum_{IA} \left(N_R^2 + N(j,k) \right) + \left(\frac{N_{IA}}{N_{OA}} \right)^2 \sum_{OA} \left(N_R^2 + N(j,k) \right) \right]^{1/2} \tag{3.6}$$

This is really nothing different than what we developed in (2.11), but now it is expressed in the detector frame rather than the telescope frame. Note that here, all values are still given in terms of e^- counts so that we may use Poisson statistics.

3.3 Comparative Photometry

In order to be able to compare the magnitudes and intensities of stars, we need a standard of measurement so that different measurements using various telescopes, CCDs, etc. will yield the same results. For this, we need a standard measure of flux (i.e. photons cm^{-2} s^{-1} or ergs cm^{-2} s^{-1}). In addition, we would like a standard set of stars to calibrate our instruments on. Later we will look in detail at the question of measurements of flux. For now, it is sufficient to assume the detector (CCD) and electronics are linear. Thus the relationship between the intensity of a star we measure in A/D units as $f(\lambda)$ and the actual flux of the star $F(\lambda)$ in photons cm^{-2} s^{-1} m^{-1} is just

$$f(\lambda) = c(\lambda)\mathcal{F}(\lambda) \tag{3.7}$$

where $c(\lambda)$ is a "constant" that depends on the specifics of our telescope, filter, CCD, A/D, etc. In general, this "constant" depends on the wavelength being measured for a variety of reasons (filter response, CCD quantum efficiency, etc.).

The magnitude scale is defined so that the difference in magnitudes is related to the log (base 10) of the ratio of fluxes as

$$m_1 - m_2 = -2.5 \log \left(\frac{\mathcal{F}_1(\lambda_1)}{\mathcal{F}_2(\lambda_2)} \right) \tag{3.8}$$

where m_1, m_2, $\mathcal{F}_1(\lambda_1)$ and $\mathcal{F}_2(\lambda_2)$ refer to the magnitudes and fluxes of two stars. We have to be careful here, though, to specify the wavelength accepted by our instrument.

If we assume both measurements are done at a fixed wavelength λ, then one can write this in terms of the measured intensity $f_1(\lambda)$, $f_2(\lambda)$ as

$$m_1 - m_2 = -2.5 \log \left(\frac{f_1(\lambda)/c(\lambda)}{f_2(\lambda/c(\lambda))} \right) = -2.5 \log \left(\frac{f_1(\lambda)}{f_2(\lambda)} \right) \tag{3.9}$$

since $c(\lambda)$ is the same in both cases. Here the assumption of fixed wavelength was *critical*. Here we have implicitly assumed that $f(\lambda)$ is the sky-subtracted signal in the language of the previous section, so that the background, sky, dark current, etc., has been subtracted.

So far, we can only get magnitude differences. What we need are stars of known flux and magnitude at given wavelengths. These are **standard stars**. If m_0 is the known magnitude of a standard star and $f_0(\lambda)$ is the measured intensity in A/D units, then the magnitude m_1 of another star whose intensity $f_1(\lambda)$ is measured at the same wavelength is

$$m_1 = m_0 - 2.5 \log \left(\frac{f_1(\lambda)}{f_0(\lambda)} \right) \tag{3.10}$$

In this way, we calibrate the measured magnitudes.

3.4 Atmospheric Considerations

Our goal is to calculate the apparent magnitude of a star as it would appear above the earth's atmosphere and to take into account the band pass and efficiencies of the whole system (filters, telescope, detector, atmosphere) so that we can compare our results to those measured by others or so they can compare their results to ours. We will use the parameters defined in Table 11.

Symbol	Meaning
$f(\lambda)$	intensity measured (in general it will depend on wavelength)
$f^*(\lambda)$	intensity that would be measured outside the earth's atmosphere
$m(\lambda)$	magnitude measured
$m^*(\lambda)$	magnitude that would be measured outside of the earth's atmosphere; typically what we are trying to solve for
$\alpha(\lambda,\theta)$	opacity of atmosphere as a function of wavelength and zenith angle, mathematically: $\ln\left[f^*(\lambda)/f(\lambda)\right]$
$\alpha_0(\lambda)$	opacity at zenith (looking straight up); essentially $\alpha(\lambda,0)$

Table 11: Parameters relevant to atmospheric corrections to photometry.

We define the **extinction coefficient** via

$$K(\lambda) = 2.5\log(e)\alpha_0(\lambda) = 1.086\alpha_0(\lambda) \tag{3.11}$$

The reason for this rather odd-looking definition will become clear soon (essentially changing from the natural base e system of α to the modified base 10 system of magnitudes). We can model the earth's atmosphere as a horizontally stratified slab so that we can relate $\alpha(\lambda,\theta)$ and $\alpha_0(\lambda)$ as follows:

$$\alpha(\lambda,\theta)X(\theta) \approx \frac{\alpha_0(\lambda)}{\cos\theta} = \alpha_0(\lambda)\sec\theta \tag{3.12}$$

where $X(\theta)$ is called the **air mass** and for angles $\theta \leq 60°$, is well approximated by $X(\theta) = \sec(\theta)$.

The air mass is the ratio of the atmosphere column density at the observation zenith angle θ to the column density at $\theta = 0$ (often referred to sea level for $\theta = 0$). The term is loosely used in the literature, unfortunately.

The relationship between the two magnitudes $m(\lambda)$ and $m^*(\lambda)$, as well as the corresponding fluxes $f(\lambda)$ and $f^*(\lambda)$ is as follows:

$$m^*(\lambda) - m(\lambda) = -2.5\log\left[f^*(\lambda)/f(\lambda)\right] \tag{3.13}$$

Since $\log\left[f^*(\lambda/f(\lambda)\right] = \log(e)\ln\left[f^*(\lambda)/f(\lambda)\right] = \log(e)\alpha(\lambda,\theta)$, we may write

$$\begin{aligned} m^*(\lambda) &= m(\lambda) - 2.5\log(e)\alpha(\lambda,\theta) && (3.14)\\ &= m(\lambda) - 2.5\log(e)\alpha_0(\lambda)X(\theta) && (3.15)\\ &= m(\lambda) - K(\lambda)X(\theta) && (3.16)\\ &= m(\lambda) - K(\lambda)\sec(\theta) && (3.17) \end{aligned}$$

whenever $\theta \leq 60°$.

Hence once we measure $m(\lambda)$ we can get $m^*(\lambda)$ if we know or can calculate $K(\lambda)$. The problem now becomes one of finding (measuring) $K(\lambda)$.

Note that we have really only determined the difference $m^*(\lambda) - m(\lambda)$, and unless we use a calibration (known) star to set the "reference level", then $m(\lambda)$ (and hence $m^*(\lambda)$) will be uncalibrated.

In Table 12, we give the "air mass" and refraction of an object versus zenith angle θ. The "air mass" includes effects due to the earth's curvature and is slightly different from $\sec(\theta)$ fo angles greater than 60°. The refraction angle assumes observations at sea level. Objects are always lower than they appear.

Now we define Z_0 to be the zenith angle (angle between the vertical and the star) as it would be measured *if there were no atmosphere present*. Then, accordingly, Z will represent the actual (measured) zenith angle of the star. Then at sea level, we have R representing $Z_0 - Z$ in arc seconds (Sorry, R is no longer the A/D gain per electron). This is essentially the correction to the measured zenith angle to get the actual zenith angle. Roughly this is given by

$$R = 58.3 \ \tan Z - 0.067 \tan^3 Z \tag{3.18}$$

θ (Degrees)	Air Mass, X	R (arc seconds)
0	1	0
10	1.02	10
20	1.06	21
30	1.15	34
40	1.30	49
50	1.55	70
60	2.00	101
70	2.90	159

Table 12: Sea level air mass and refraction versus zenith angle.

A plot of $m(\lambda)$ versus $X(\theta)$ measured over time as a star rises or sets *should* be a straight line if the atmosphere is stable over this time. Since $m(\lambda) = m^*(\lambda) + K(\lambda)\cos\theta$, the slope of the line would be $K(\lambda)$ and the zero intercept would be $m^*(\lambda)$, which is the extra atmospheric magnitude we are trying to measure.

Note that, in theory, if we measure $m(\lambda)$ for the same star at two air masses, we can hen determine $m^*(\lambda)$ and $K(\lambda)$. Conversely, if we know $m^*(\lambda)$ (from standard stars) we can determine $K(\lambda)$. Note that we can measure $K(\lambda)$, but as stated before, we really only measure magnitude differences (i.e. $m^*(\lambda) - m(\lambda)$) unless we calibrate our magnitude scale using a standard star.

In Table 13, we list the extinction coefficient and transmission versus wavelength using a "standard" sea level atmosphere assuming the zenith angle is zero ($\theta = 0$). By definition, in this case the air mass is $X(\theta = 0) = 1$. The extinction coefficient unit of measure is "magnitudes".

λ (microns)	$K(\lambda)$ (mag)	Transmission (%)
0.30	4.89	1.1
0.32	1.41	27.3
0.34	0.91	43.0
0.36	0.74	51.0
0.38	0.60	58.0
0.40	0.50	63.0
0.45	0.34	73.0
0.50	0.25	79.0
0.55	0.21	82.0
0.60	0.19	84.0
0.65	0.14	88.0
0.70	0.10	91.1
0.80	0.07	93.9
0.90	0.05	95.3
1.00	0.04	96.2
1.20	0.03	97.2
1.40	0.02	97.9
1.60	0.02	98.3
1.80	0.02	98.5
2.00	0.01	98.7

Table 13: Extinction coefficient and transmission as a function of wavelength assuming zenith viewing at sea level with a "standard" atmosphere.

3.4.1 Finding $m^*(\lambda)$ and $K(\lambda)$

If we measure the magnitude of a star for two different air masses, we can solve for $m^*(\lambda)$ and $K(\lambda)$ as follows. First, let $m_1(\lambda)$ and $X_1(\theta_1)$ be measured at angle θ_1. Similarly, $m_2(\lambda)$ and $X_2(\theta_2)$ are measured at angle θ_2. Then as before, we have

$$m_1(\lambda) = m^*(\lambda) + K(\lambda)X_1(\theta_1) \tag{3.19}$$

$$m_2(\lambda) = m^*(\lambda) + K(\lambda)X_2(\theta_2) \tag{3.20}$$

Then the extra-atmosphere magnitude and extinction coefficient can be be obtained as

$$m^*(\lambda) = \frac{m_1(\lambda)X_2(\theta_2) - m_2(\lambda)X_1(\theta_1)}{X_2(\theta_2) - X_1(\theta_1)} \tag{3.21}$$

$$K(\lambda) = \frac{m_2(\lambda) - m_1(\lambda)}{X_2(\theta_2) - X_1(\theta_1)} \tag{3.22}$$

The primary disadvantage to this method is that it assumes the atmosphere is stable over the time it takes for the star to go from θ_1 to θ_2. Usually it is desirable to have at least a 30° difference between θ_1 and θ_2 to give reasonable accuracy for $m^*(\lambda)$ and $K(\lambda)$. In theory, the measured $K(\lambda)$ could now be used for other stars to find $m^*(\lambda)$ as long as the atmosphere is stable.

Another way of determining $K(\lambda)$ is to measure two or more known stars of the same spectral class at significantly different air masses using the same filter(s). Since in this case, we know $m^*(\lambda)$ for each star, we have

$$m_1(\lambda) = m_1^*(\lambda) + K(\lambda)X_1(\theta_1) \tag{3.23}$$

$$m_2(\lambda) = m_2^*(\lambda) + K(\lambda)X_2(\theta_2) \tag{3.24}$$

Since we specified the same filter is used for each observation, we get the extinction coefficient to be

$$K(\lambda) = \frac{m_1(\lambda) - m_2(\lambda) - (m_1^*(\lambda) - m_2^*(\lambda))}{X_1(\theta_1) - X_2(\theta_2)} \tag{3.25}$$

By using stars of the same spectral class, we minimize any mismatch problems our filters may have. Also by writing $K(\lambda)$ as involving only the differences in magnitudes $m_1(\lambda) - m_2(\lambda)$ eliminates the need to calibrate the measured magnitudes $m_1(\lambda)$ and $m_2(\lambda)$.

3.4.2 Finding the absolute flux of a star

To find the absolute flux of a star, we need to know the response of tall of the elements of our system including telescope, filters, detector, sky background and atmospheric opacity. We define these responsivities quantitatively as given in Table 14.

Symbol	Meaning
$f(\lambda)$	measured star intensity in A/D units
$F^*(\lambda)$	actual [specific] star flux above atmosphere in photons $\mathrm{cm^{-2}\,s^{-1}\,m^{-1}}$
$F(\lambda)$	[specific] star flux at telescope aperture
$\varepsilon(\lambda)$	optical efficiency, including telescope, filter, glass, etc. (fraction of photons entering telescope aperture that make it to detector)
$QE(\lambda)$	quantum efficiency of CCD in e^-/photons
A	effective aperture area of telescope in cm^2
$F_B(\lambda)$	emitted sky background in photons $\mathrm{cm^{-2}\,s^{-1}\,steradian^{-1}\,m^{-1}}$
R	CCD response (A/D counts per e^-)
R_0	A/D no signal value (offset)
$\alpha(\lambda, \theta)$	atmospheric opacity. Depends on λ and zenith angle of observation. $\alpha(\lambda, \theta) = \ln[F^*(\lambda)/F(\lambda)]$
i_{DC}	CCD dark current in e^-/s
τ_{DC}	integration time in seconds
$\Omega(\lambda)$	solid angle per CCD pixel in steradians
$\Delta\lambda$	optical bandpass of system (filter) in m

Table 14: Variables of use in this section.

For convenience, we define the **effective area** of the telescope via

$$A_\varepsilon(\lambda) = A\varepsilon(\lambda)QE(\lambda)\Delta\lambda \tag{3.26}$$

which is similar to our discussion in Section 2.2 (though note that it actually has units of volume due to the presence of the bandwidth). Then the measured star intensity in A/D units is

$$f(\lambda) = \int [F(\lambda)A\varepsilon(\lambda)QE(\lambda) + F_B(\lambda)A\varepsilon(\lambda)QE(\lambda)\Omega(\lambda)]\, d\lambda\, dt \tag{3.27}$$

$$\approx \left[F^*(\lambda)e^{-\alpha(\lambda,\theta)}A\varepsilon(\lambda)QE(\lambda)\Delta\lambda + F_B(\lambda)\Omega(\lambda)A\varepsilon(\lambda)QE(\lambda)\Delta\lambda + i_{\mathrm{DC}}\right]\tau R + R_0 \tag{3.28}$$

This is a bit ugly (and also a bit heuristic), so we'd like to rewrite it in terms of measured quantities. Defining $f_B(\lambda) \equiv [F_B(\lambda)\Omega(\lambda)A\varepsilon(\lambda)\Delta\lambda + i_{\mathrm{DC}}]\, R\tau + R_0$ (essentially a noise flux), we may rewrite (3.28) as

$$f(\lambda) = F^*(\lambda)e^{-\alpha(\lambda,\theta)}A_\varepsilon(\lambda)\tau R + f_B(\lambda) \tag{3.29}$$

The first term is the signal from the source, whereas the rest is due to different noise sources. Solving for the intrinsic flux, we have

$$F^*(\lambda) = \frac{e^{\alpha(\lambda,\theta)}\,[f(\lambda) - f_B(\lambda)]}{A_\varepsilon(\lambda)\tau} \tag{3.30}$$

Notice that $f_B(\lambda)$ is precisely the value that the same pixel would have if there were no star present (i.e., if we were only measuring the background and dark current). We can easily get $f_B(\lambda)$ by making another measurement of the same integration time of a blank field (same zenith angle approximately or by using a nearby pixel value which should be equivalent (assuming flat fielding was done first).

Notice that $A_\varepsilon(\lambda)$ is only a function of system parameters and does not depend on the atmosphere. In theory, we need only determine $A_\varepsilon(\lambda)$ once for each filter used and it should be consistent thereafter. This assumes that the CCD is stable from one observation to the next.

Since a star will usually deposit photons in more than one pixel we should sum over all pixels that have significant star light. We then write $F^*(\lambda)$ as

$$F^*(\lambda) = \frac{e^{\alpha(\lambda,\theta)}}{A_\varepsilon(\lambda)R\tau}\sum [f(\lambda) - f_B(\lambda)] \tag{3.31}$$

In Section 3.1, we calculated the total number of electrons n produced in the CCD associated with the star as

$$n = \frac{1}{R}\sum [f(\lambda) - f_B(\lambda)] \tag{3.32}$$

So we may rewrite (3.31) as

$$F^*(\lambda) = \frac{e^{\alpha(\lambda,\theta)}}{A_\varepsilon(\lambda)\tau}n \tag{3.33}$$

3.5 Filters

Note: A lot of this material is adapted from Carroll & Ostlie's An Introduction to Modern Astrophysics, 2nd Edition.

3.5.1 Bolometric Magnitude

In measuring photometry, we are often trying to measure the amount of electromagnetic flux incident on a detector. However, there is no detector that is completely sensitive in all wavelengths (such a detector would be called a **perfect bolometer**). In fact, no detectors exist that can measure flux even poorly in all wavelengths. Instead, they are all limited to some (typically small) subset of the EM spectrum. So far, though, when we've talked about magnitudes, we've typically only been talking about **bolometric magnitudes** (unless the magnitude was denoted as $m(\lambda)$, where wavelength dependence was made explicit). The bolometric magnitude is what *would* be measured by a perfect bolometer if there were no losses due to quantum inefficiencies, atmospheric extinction, interstellar reddening, etc. In terms of the specific flux, F_λ (sometimes called the **spectral energy distribution**, or **SED**) of an object and your magnitude system's zero point, the bolometric magnitude is defined as

$$m_{\mathrm{bol}} = -2.5\log_{10}\left(\int d\lambda F_\lambda\right) + m_{\mathrm{bol},0} \tag{3.34}$$

We keep the integral over all wavelengths of the specific flux in there as a pedantic gesture to illustrate the difference between the bolometric magnitude from the other magnitudes we will be discussing. Note, though that $\int d\lambda\, F_\lambda$ is simply the overall flux F of the object. We may define other fluxes, like the visible flux via

$$F_{\mathrm{Visible}} = \int_{\lambda_1}^{\lambda_2} d\lambda\, F_\lambda \tag{3.35}$$

where λ_1 and λ_2 are chosen as the limits of the visible spectrum (say 350 nm and 750 nm, or thereabouts).

Since we have no hope of directly measuring the bolometric magnitude of an object (even if we go to space, etc.), we sidestep the problem by making sets of filters that only allow certain **bands** (intervals) of wavelengths through, and at well-known efficiencies. In addition to knowing just what's able to be measured by our detector, we also get an idea of what the "color" of an object is.

3.5.2 The Johnson-Morgan Filter System

The classic system of filters that we will discuss is the Johnson-Morgan (sometimes the name Cousins is thrown in here, too) of filters. While there are a great many filters in this system, we will look at the main three, U, B, and V. Some basic information about these filters is shown in Table 15.

Symbol	"Color"	Central Wavelength	FWHM
U	Ultraviolet	365 nm	68 nm
B	Blue	445 nm	98 nm
V	Visual	550 nm	89 nm

Table 15: Basic filters of the Johnson-Morgan system.

With each filter, we can determine a magnitude in that filter. For instance, we are able to determine the B-band magnitude by measuring the magnitude of an object with a B filter on the telescope. Each filter will let in different fluxes for a given object, so a zero-point magnitude must be determined for each filter. For instance, we may define a guide star to be at magnitude zero in *all* bands, so its flux sets the zero point for the magnitude in a given band. The responses for the U, B, V, R, and I filters are shown in Figure 4

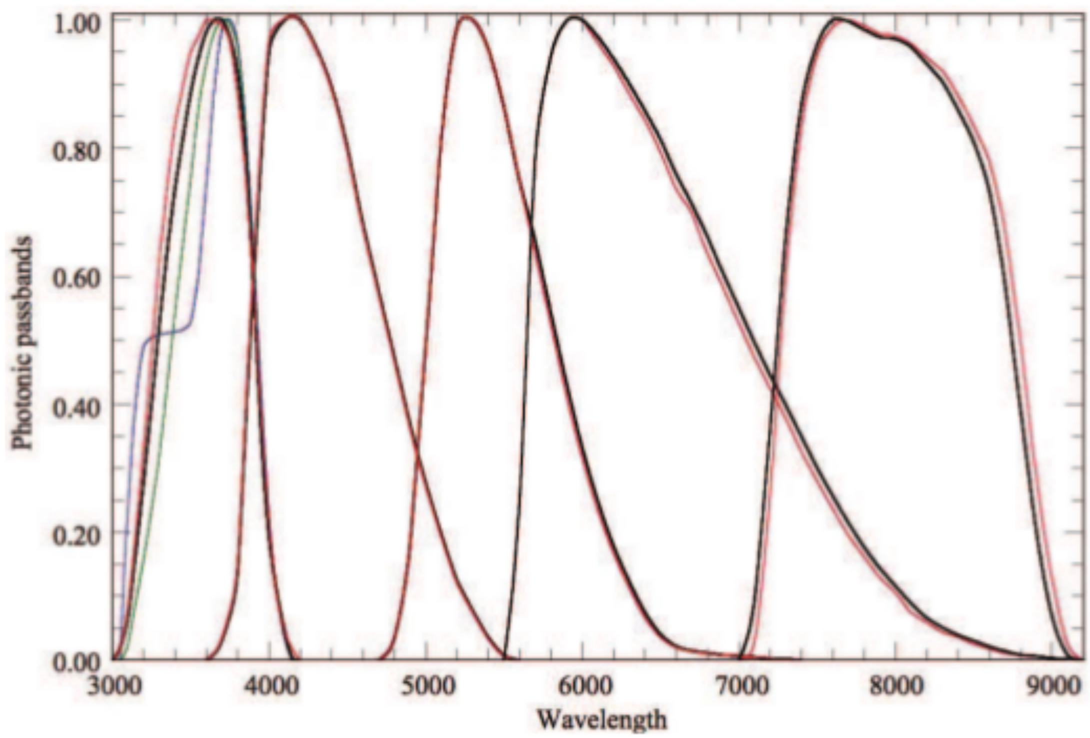

Figure 4: Sensitivity functions of the Johnson filters (black lines). From left to right, they are U, B, V, R, and I.

In addition to the Johnson system of filters, different observatories and astronomers use different systems to suit their needs. A popular system nowadays is the SDSS *ugriz* system. Sometimes you'll see u', g', r', i', and z' to label these filters as well. The primes *do* mean something, as these system are different in nature. These filters were named after the project where they were first thoroughly used, the Sloan Digital Sky Survey (SDSS). Their responses are shown in Figure 5.

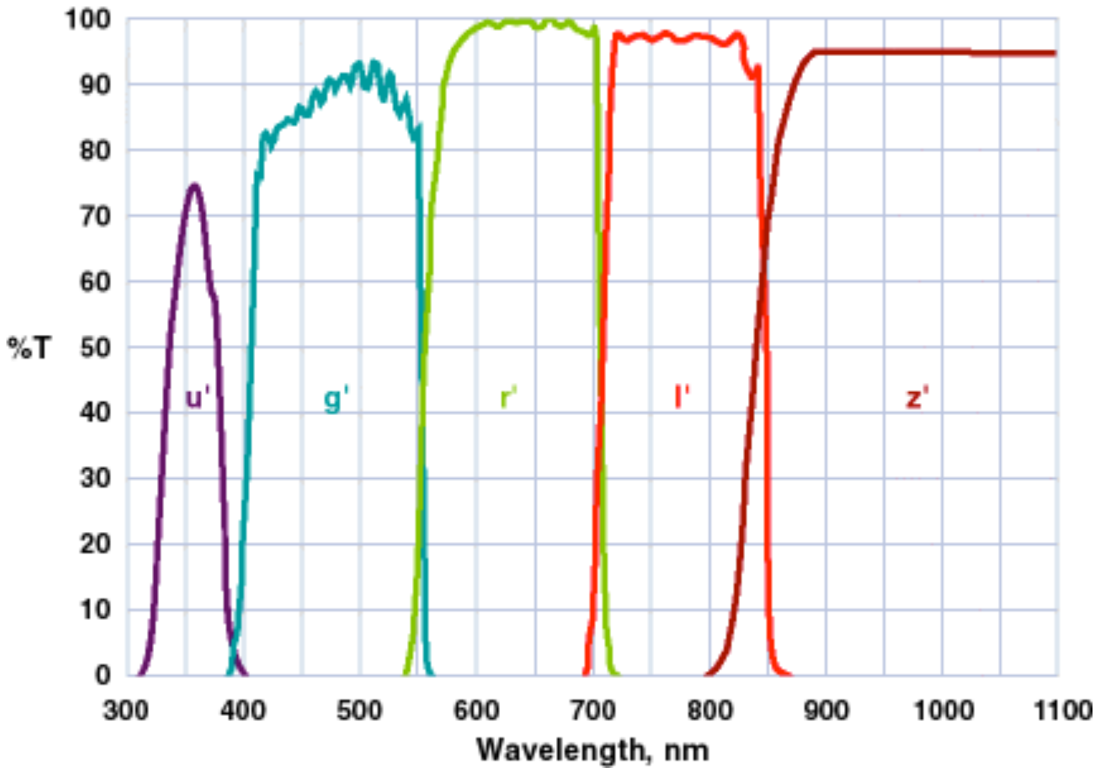

Figure 5: Sensitivity functions of the SDSS $u'g'r'i'z'$ filters.

3.5.3 Color Indices and Corrections

When multiple exposures of an object are taken in different filters, we gain a wealth of information. Not only do we obtain the magnitudes in multiple bands, but the *differences* in the various magnitudes tell us about the relative color of the object. We define the **color index** of an object in two filters by the difference in magnitude of that object as measured in the two filters. For instance, the $B-V$ color index of an object is given by

$$B - V \equiv M_B - M_V = m_B - m_V \tag{3.36}$$

Note that we have denoted the absolute and apparent magnitudes with subscripts indicating which filter they correspond to. Sometimes in the literature, we see just U to represent the U-band magnitude (apparent). We shall avoid such notation here, but it is quite common to see the apparent magnitude in a filter to just be represented by the filter symbol.

Another thing to note is that to get a color, *we don't need absolute magnitudes.* This is where the niceties of the logarithmic magnitude system become apparent. We need only look at the differences of magnitudes to find the color. No distances need to be known (assuming any interstellar reddening or redshift is negligible or at least accountable). Quite often knowing the color of a star is *just as important if not more so* than knowing the actual brightness. For a blackbody, for instance, the color is directly related to the temperature of the object. As such, quite often HR diagrams (typically a plot of the luminosity against its effective temperature) will be represented with a color on the x-axis and an absolute magnitude (or at least a distance-normalized magnitude) on the y-axis. This is more the "observer" picture of an HR diagram, whereas the more traditional $L - T_{\rm eff}$ diagram is more of a "theorist" view. This is because we measure filter magnitudes (and thus colors) directly, and we *infer* luminosities and temperatures.

Despite its impossibility of being directly measured, we still would *like* to determine an object's bolometric magnitude. For objects with known spectra (F_λ), often we have a **bolometric correction** available. The bolometric correction of an object is the quantity that needs to be added to the visual (V-band) magnitude to get what the bolometric magnitude would be. Recall that if we have the spectrum, we can deduce what the bolometric magnitude would be (if we already know the distance and size). We'll figure out how to use this information in just a moment. Mathematically, the bolometric correction is

$$BC \equiv m_{\rm bol} - m_V = M_{\rm bol} - M_V \tag{3.37}$$

Astronomers have large tables that give pre-calculated bolometric corrections for stars of various spectral classes. In general, though, finding the bolometric correction is not an obvious task.

Example: Color Indices and Bolometric Corrections Sirius, the brightest-appearing star in the sky, has U, B, and V magnitudes of $m_U = -1.47$, $m_B = -1.43$, and $m_V = -1.44$. Thus for Sirius,

$$U - B = -1.47 - (-1.43) = -0.04 \tag{3.38}$$

and

$$B - V = -1.43 - (-1.44) = 0.01 \tag{3.39}$$

The bolometric correction for Sirius is $BC = -0.09$, so its apparent bolometric magnitude is

$$m_{\rm bol} = m_V + BC = -1.44 + (-0.09) = -1.53 \tag{3.40}$$

To perform such a calculations, and many like them, we must first talk about **sensitivity functions**. Sometimes these are called the response function, the transmission function, or any number of things. The idea, though, is that the sensitivity function of a filter determines what fraction of photons of a given wavelength pass through the filter to a detector. We already saw these in Figures 4 and 5. It's important to notice that these are very dependent on wavelength, especially at the fringes of sensitivity. We will denote the sensitivity of the ith filter (no filter in particular) as $\mathcal{S}_i(\lambda)$. $S_i(\lambda)$ is always between 0 and 1. When $\mathcal{S}_i(\lambda) = 0$, the the filter is opaque to that wavelength, and if $\mathcal{S}_i(\lambda) = 1$, then the filter is transparent to that wavelength. For the purposes of this discussion we will be neglecting attenuation due to interstellar reddening, the atmosphere, and intrinsic inefficiencies in the telescope/CCD.

With this machinery, we may determine what the flux through any given filter could be, in a fashion similar to (3.35). Through a given filter i, the flux through that filter from an object with specific flux F_λ would be

$$F_i = \int d\lambda \, S_i(\lambda) F_\lambda \tag{3.41}$$

Note how the flux is attenuated by the sensitivity function, and so outside of the region of sensitivity of the filter, F_λ is chopped to zero by $S_i(\lambda)$. We might think that the sensitivity function of a perfect bolometer would be $S_{\text{perfect}} = 1$, so that there is 100% transmission at all wavelengths. Correspondingly, the magnitude that would be measured in that filter would be

$$m_i = -2.5 \log F_i + m_{i,0} \tag{3.42}$$

where, again, $m_{i,0}$ is the zero point in that filter. Now we can use this sort of thinking to come up with a more rigorous definition for color indices. For instance, the "formula" for $U - B$ of an object would be

$$U - B = -2.5 \log \left(\frac{\int d\lambda \, S_U(\lambda) F_\lambda}{\int d\lambda \, S_B(\lambda) F_\lambda} \right) + C_{U-B} \tag{3.43}$$

where C_{U-B} is simply the difference in the two zero points, $C_U - C_B$. So we see now that if we know F_λ and the magnitude in *any* filter, we know it in *all* of them. However, we typically don't know F_λ to good enough precision to be happy with just one filter (and often we don't know it at all, since spectroscopy is harder than photometry), so we typically have good filter coverage to minimize the errors.

We now can return back to how we calculate bolometric corrections, which is now a trivial exercise. A bolometric correction is nothing more than a color index with one filter being that of a perfect bolometer ($S(\lambda) = 1$):

$$m_{\text{bol}} - m_V = -2.5 \log \left(\frac{\int d\lambda F_\lambda}{\int d\lambda \, S_V(\lambda)} \right) + C_{\text{bol}-V} = BC \tag{3.44}$$

As a cultural aside, C_{bol} was not chosen in the same way that the other C_i's were (at least, not originally). Astronomers wanted the bolometric correction to always be negative (with the reasoning that integrating over *all* wavelengths should "be brighter" than only a subset). Eventually a value was chosen for C_{bol}, but afterwards supergiants were discovered that have positive bolometric corrections. However, the damage was done, and now the system is well in place.

3.5.4 Photometric Redshift

We can squeeze another use out of the various colors that filters provide us. In the case of a known spectral energy distribution (SED, same as specific flux, F_λ), we can, in theory, calculate what the expected color indices would be. However, for redshifted objects (typically extragalactic objects), the SED will be altered via $\lambda \to (1 + z)\lambda$. As a result, the measured color indices will be different.

We can use this effect to estimate redshifts (and thus via Hubble's law, distances) to objects. One could simply dial z up from 0 until the difference between the new, redshifted color indices and the observed color indices reaches a minimum. This technique is called a **photometric redshift**. We call it that in contrast to spectroscopic redshifts, where are obtained by seeing how far known absorption or emission features are moved in a spectrum.

Photometric redshifts are sort of a "poor man's redshift" because they are often quite imprecise, with uncertainties of up to $\delta z = 0.5$ not uncommon. Interstellar reddening, both from the host as well as the Milky Way also act to muddle this process up, but it is still a good first-order guess to get a distance to an object when ample telescope time to "do it right" with a spectrometer is not available.

3.5.5 Interstellar Reddening and Color Excess

Being so distant, objects are often reddened by **interstellar reddening**, which we've already mentioned, but not defined. Dust in between stars acts to scatter photons, but it prefers short-wavelength photons. This is the exact same reason why sunsets are red and the sky is blue: the blue photons from someone else's sunset are scattered into our sky, leaving their sunset red. The same thing happens to stellar objects whose light have a long way to travel (even in our own galaxy).

We define the **total extinction** in a filter as the change in magnitude (in that filter) that is caused by interstellar reddening. Typically it is denoted by $A(i)$ for the ith filter. In equation form, we have

$$m_{i,\mathrm{obs}} = m_{i,\mathrm{intrinsic}} + A(i) \tag{3.45}$$

Not only will this extinction cause a decreased incident flux here at Earth, but since it's *reddening*, it will cause different extinctions in different filters. Since different extinctions cause different changes in magnitudes, the color indices of an object are affected by interstellar reddening. See Figure 6 to see how some local galaxies cause extinction of light in various wavelengths.

This differential extinction gives rise to the definition of a **color excess**. For convenience, we'll define it in terms of the B and V filters, but the same idea applies to any color index:

$$E(B-V) \equiv (B-V)_{\mathrm{observed}} - (B-V)_{\mathrm{intrinsic}} = A(B) - A(V) \tag{3.46}$$

The color excess of an object is really more of a property of the medium between the observer and the source more so than the source itself, so we can act to mitigate its effects. For instance, when viewing distant supernovae, we may know something about its host galaxy (how dusty it is, etc.), so we can estimate a value for $E(B-V)_{\mathrm{host}}$. Additionally, if we know what part of the Milky Way we're looking through, we can also probably come up with some value $E(B-V)_{\mathrm{MW}}$ with which to correct the incoming light.

However, for very distant objects, their redshift can complicate this process. For instance, the light that was in the B and V bands when it was emitted was reddened by the host galaxy dust just as we would expect. However, along the way, the photons are redshifted as they reach the Milky Way. Now the $E(B-V)_{\mathrm{MW}}$ is acting on light that was emitted at a shorter wavelength than it is now (and the B and V-band photons that were reddened by their host are now entering the Milky Way at longer wavelengths). The light that is now entering the Milky Way probably started its journey at a shorter wavelength, and was reddened by its host galaxy, but not in the same way as the B and V light was since the extinction acts differently at different wavelengths and at different places. You can see how this quickly gets convoluted.

3.5.6 K-Corrections

Not only is the business of keeping track of color excess from distant sources difficult, but the entire photometric system is now totally bonkers. The magnitudes you are measuring in each filter are

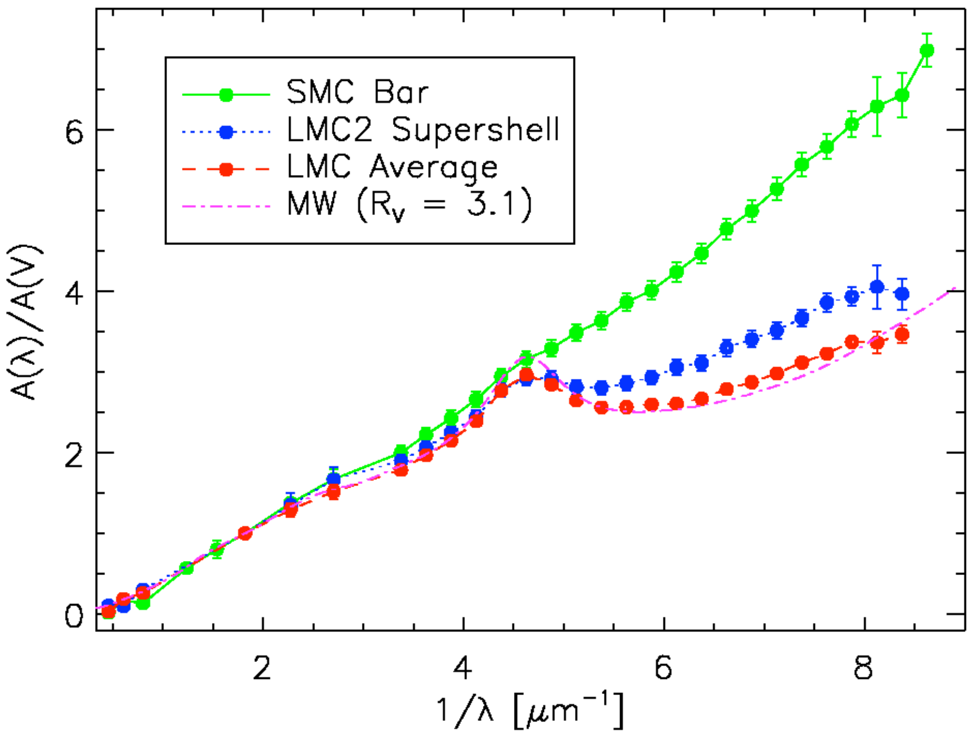

Figure 6: Extinction curves for the Milky Way and the Magellanic Clouds. Note that they are wavelength dependent, and they even vary depending on where you are observing through.

no longer representative of the actual color of the object (as we've already mentioned regarding photometric redshifts). While *observing* in the bands is completely fine, we can't say much about the actual source we are investigating because the magnitudes we record are nearly meaningless. This is because the λ's in $S(\lambda)$ and F_λ *are no longer the same.*

In the SED, F_λ, the wavelengths described are those *emitted* by the source. However, $S(\lambda)$ doesn't "know" about that. Instead, it just deals with the wavelengths it *receives*. For nearby objects, where the wavelengths of photons don't change along the path from the source to the observer, this isn't a problem, but for substantially redshifted objects, this poses a huge problem in getting accurate photometry. See Figure 7 for an example of how nasty this can get.

Astronomers have developed a way to fix this, though. The ***K*-Correction** is the difference between the source's rest-frame photometry and the observer's rest-frame photometry. Mathematically, we have

$$m_{j,\text{observed}} = m_{i,\text{rest}} + K_{ij} \tag{3.47}$$

Here, K_{ij} is the K-correction that converts observed magnitudes in the rest-frame i-band and

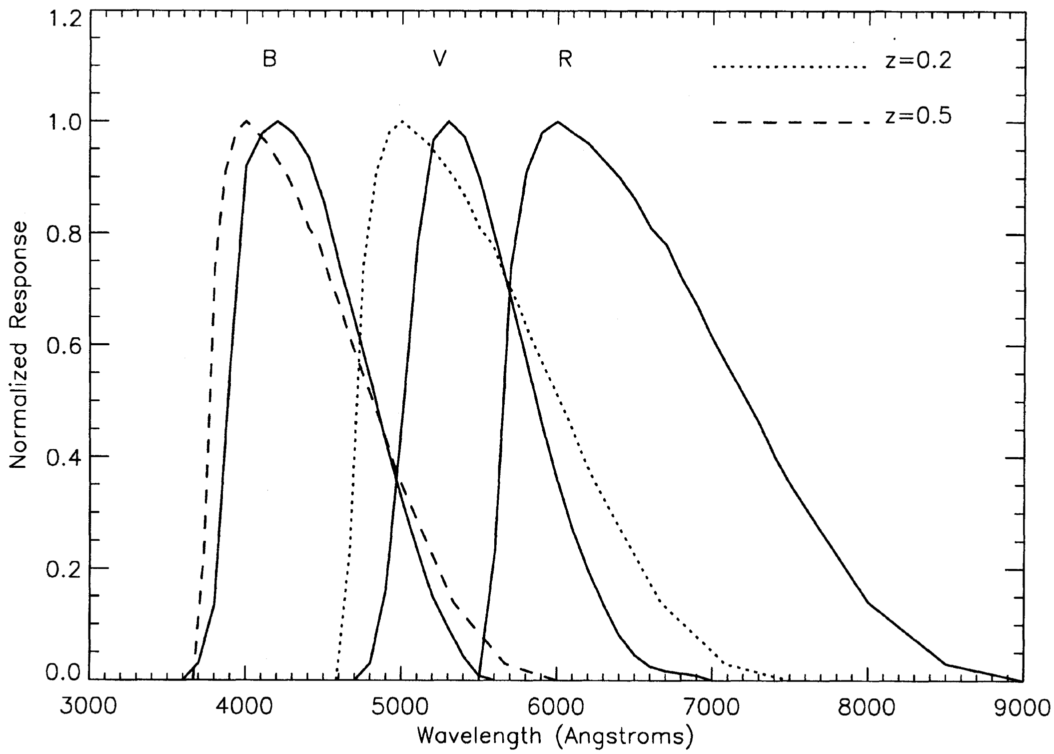

Figure 7: "Blueshifted" sensitivity functions of the R band at different redshifts. The solid lines show the standard rest frame sensitivity functions of the B, V, and R filters. The dotted line is showing the sensitivity as a function of the source's rest frame wavelengths at $z = 0.2$, and the dashed line is the same for $z = 0.5$. In those cases, the filter is pulling in photons that are closer to V- and B-band filters, respectively. From Kim et al. 1996.

converts them to their corresponding magnitudes in the observer's frame j-band. Mathematically, though, this is a bit more complicated of a correction than our previous color indices and bolometric corrections. We must account for the difference in zero points of the two filters, as well as the redshifted SED, and finally, the reduced intensity of redshifted light. The formula can be expressed in two ways. First, we'll investigate the more straightforward one:

$$K_{ij} = \underbrace{-2.5\log\left(\frac{\int d\lambda\, \mathcal{Z}(\lambda)S_i(\lambda)}{\int d\lambda\, \mathcal{Z}(\lambda)S_j(\lambda)}\right)}_{\text{Zero Point Correction}} + \underbrace{2.5\log(1+z)}_{\text{Intensity Correction}} \underbrace{+2.5\log\left(\frac{\int d\lambda\, F(\lambda)S_i(\lambda)}{\int d\lambda F(\lambda/(1+z))S_j(\lambda)}\right)}_{\text{Wavelength Correction}} \tag{3.48}$$

Here we've labeled the three terms by what their responsibilities are. If $z \to 0$, this becomes a simple color index, with the first term simply being the difference in zero points. If $i = j$, then the whole correction vanishes, as it just maps an unredshifted magnitude in a band to the exact same unredshifted band.

An alternate way to look at this is to combine the last two terms into one:

$$K_{ij} = -2.5\log\left(\frac{\int d\lambda\, \mathcal{Z}(\lambda)S_i(\lambda)}{\int d\lambda\, \mathcal{Z}(\lambda)S_j(\lambda)}\right) + 2.5\log\left(\frac{d\lambda\, F(\lambda)S_i(\lambda)}{\int d\lambda'\, F(\lambda')S_j(\lambda'(1+z))}\right) \tag{3.49}$$

Here we've just combined the redshift logarithm term with the last term, then done a change of variables on the bottom integral from λ to $\lambda' = \lambda/(1+z)$. This has a physical interpretation as well. The first term still just corrects for zero points between filters, but the second term now is the correction for the *unredshifted* source photons passing through a *blueshifted* filter. In this case, the SED is the same in both situations (emission and collection), but the filter is now sensitive to much smaller wavelengths. This is essentially the idea presented in Figure 7. The various re-plottings of the R-band are at new "blueshifts".

The first presentation does a good job of showing the physics of what's happening to the photons as they make their journey, but the second presentation shows more what you're *getting* with the observed filter. Regardless, they both give the same result (obviously), and no one ever really uses these integrals by hand, since the sensitivity functions are never analytic.

With a good K-correction, we can convert observed measurements in any filter to rest-frame measurements in any filter. However, to minimize error (due to not knowing F_λ precisely), we should match the blueshifted observing filter to the closest available rest filter (or alternatively, match the redshifted rest filter to the closest observed filter). In this way, the impact of the last term is kept to a minimum, putting most of the work on the zero point correction, which is presumably well-known. This is most easily seen in (3.49). We are looking for the filters i and j where $S_i(\lambda) = S_j(\lambda(1+z))$ so that the term can simply drop out.