Chapter 21 – Electric Charge

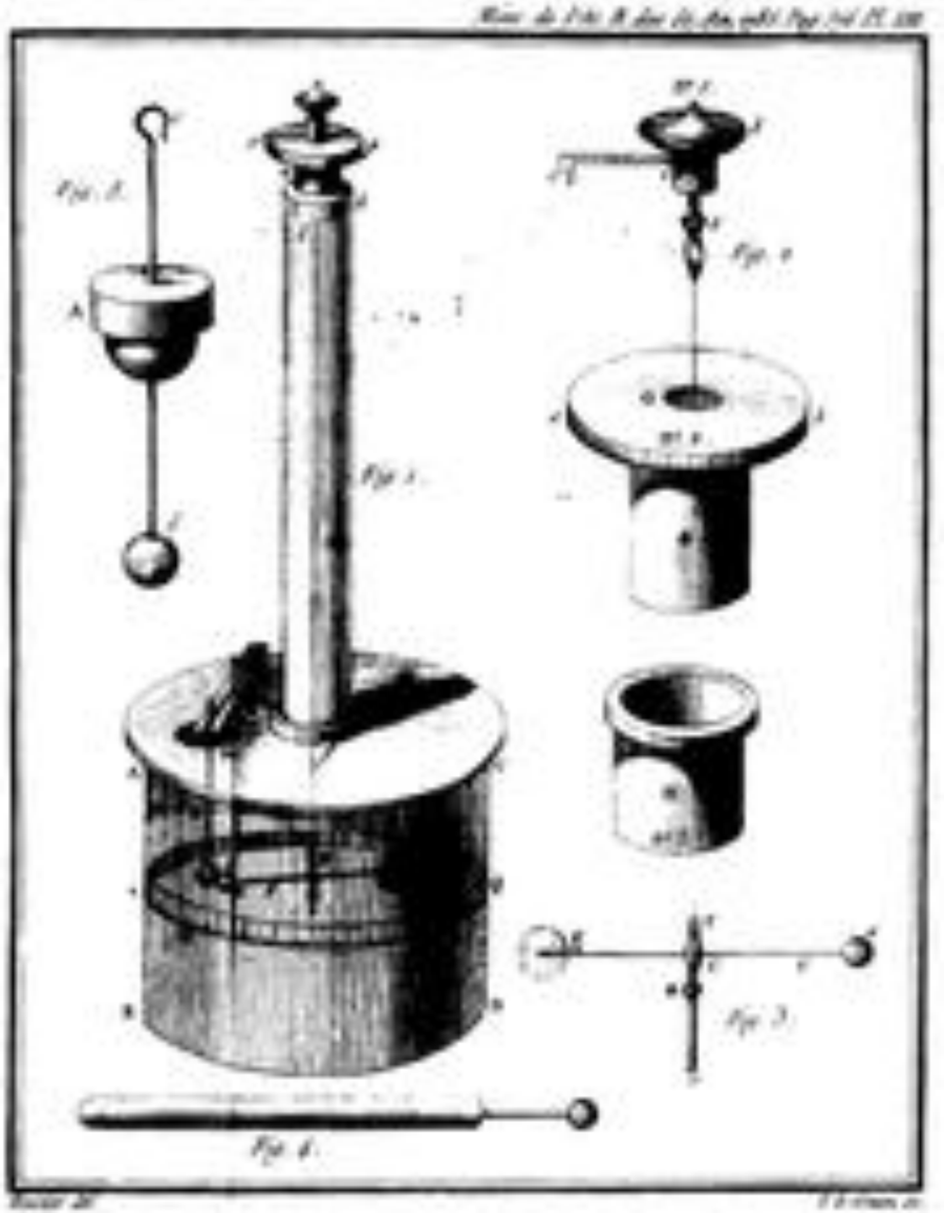
- Historically people knew of electrostatic effects
- Hair attracted to amber rubbed on clothes
- People could generate "sparks"
- Recorded in ancient Greek history
- 600 BC Thales of Miletus notes effects
- 1600 AD - William Gilbert coins Latin term *electricus* from Greek *ηλεκτρον* (*elektron*) – Greek term for Amber
- 1660 Otto von Guericke – builds electrostatic generator
- 1675 Robert Boyle – show charge effects work in vacuum
- 1729 Stephen Gray – discusses insulators and conductors
- 1730 C. F. du Fay – proposes two types of charges – can cancel
- Glass rubbed with silk – glass charged with "*vitreous electricity*"
- Amber rubbed with fur – Amber charged with *"resinous electricity"*

A little more history

- 1750 Ben Franklin proposes “vitreous” and “resinous” electricity are the same ‘electricity fluid” under different “pressures”
- He labels them “positive” and “negative” electricity
- Proposaes “conservation of charge”
- June 15 1752(?) Franklin flies kite and “collects” electricity
- 1839 Michael Faraday proposes “electricity” is all from two opposite types of “charges”
- We call “positive” the charge left on glass rubbed with silk
- Today we would say ‘electrons” are rubbed off the glass

Torsion Balance

- Charles-Augustin de Coulomb - 1777



Used to measure force from electric charges and to measure force from gravity

$\tau = -\kappa\theta$ - "Hooks law" for fibers

(recall F = -kx for springs)

General Equation with damping

θ – angle

I – moment of inertia

C – damping coefficient

κ – torsion constant

τ – driving torque

$$I\frac{d^2\theta}{dt^2} + C\frac{d\theta}{dt} + \kappa\theta = \tau(t)$$

Solutions to the damped torsion balance

$$\theta = Ae^{-\alpha t}\cos(\omega t + \phi)$$

General solutions are damped oscillating terms – ie damped SHO
A = amplitude
t = time
α = damping frequency = 1/damping time (e folding time)
ϕ = phase shift
ω = resonant angular frequency

$$\alpha = C/2I$$

If we assume a lightly damped system where:

$$C \lll \sqrt{\kappa I}$$

Then the resonant frequency is just the undamped resonant frequency

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\kappa/I}$$

$\omega_n = \sqrt{(\kappa/I)}$ (ω_n = "natural undamped resonant freq")
recall for a spring with mass m that
$\omega = \sqrt{(k/m)}$ where k=spring constant

General solution with damping

- If we do NOT assume small damping then the resonant freq is shifted DOWN
- From the “natural undamped resonant freq:
- $\omega_n = \sqrt{(\kappa/I)}$
- Note the frequency is always shifted DOWN

$$\omega = \sqrt{\omega_n^2 - \alpha^2} = \sqrt{\kappa/I - (C/2I)^2}$$

Constant force and critical damping

- When the applied torque (force) is constant
- The drive term $\tau(t)$ = F*L where F is the force
- L= moment arm length

$$\theta = FL/\kappa$$

We want to measure the force F

To do this we need $\boldsymbol{\kappa}$

We get $\boldsymbol{\kappa}$ from measuring the resonant freq $\boldsymbol{\omega}$

Then $\underline{\boldsymbol{\kappa = \omega^2 I}}$

In real torsion balances the system will oscillate at resonance and we want to damp this

Critical damping (fastest damping) for $C_c = 2\sqrt{\kappa I}$

Sinusoidal Driven Osc

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = F_0\sin(\omega t) \ ,$$

$$x(t) = \frac{F_0}{Z_m\omega}\sin(\omega t + \phi)$$

$$\phi = \arctan\left(\frac{2\omega\omega_0\zeta}{\omega^2 - \omega_0^2}\right)$$

- Max amplitude is achieved at resonance
- $\omega_r = \omega_0\sqrt{(1-2\zeta^2)}$
- For a mass and spring $\omega_0 = \sqrt{(k/m)}$
- Normal damping term $\Gamma = 2\zeta\omega_0$

Universal Normalized (Master) Oscillator Eq

No driving (forcing) function equation

System is normalized so undamped resonant freq $\omega_0 = 1$. $\tau = t/t_c$ t_c = undamped period $\omega = \omega/\omega_0$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\zeta \frac{\mathrm{d}q}{\mathrm{d}\tau} + q = 0$$

With sinusoidal driving function

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\zeta \frac{\mathrm{d}q}{\mathrm{d}\tau} + q = \cos(\omega\tau).$$

We will consider two general cases

Transient $q_t(t)$ and steady state $q_s(t)$

Transient Solution

$$q_t(\tau) = \begin{cases} e^{-\zeta\tau}\left(c_1 e^{\tau\sqrt{\zeta^2-1}} + c_2 e^{-\tau\sqrt{\zeta^2-1}}\right) & \zeta > 1 \text{ (overdamping)} \\ e^{-\zeta\tau}(c_1 + c_2\tau) = e^{-\tau}(c_1 + c_2\tau) & \zeta = 1 \text{ (critical damping)} \\ e^{-\zeta\tau}\left[c_1 \cos\left(\sqrt{1-\zeta^2}\tau\right) + c_2 \sin\left(\sqrt{1-\zeta^2}\tau\right)\right] & \zeta < 1 \text{ (underdamping)} \end{cases}$$

Steady State Solution

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\zeta \frac{\mathrm{d}q}{\mathrm{d}\tau} + q = \cos(\omega\tau) + i\sin(\omega\tau) = e^{i\omega\tau}.$$

$$q_s(\tau) = Ae^{i(\omega\tau+\phi)}.$$

$$q_s = Ae^{i(\omega\tau+\phi)},\ \frac{\mathrm{d}q_s}{\mathrm{d}\tau} = i\omega Ae^{i(\omega\tau+\phi)},\ \frac{\mathrm{d}^2 q_s}{\mathrm{d}\tau^2} = -\omega^2 Ae^{i(\omega\tau+\phi)}.$$

$$-\omega^2 Ae^{i(\omega\tau+\phi)} + 2\zeta i\omega Ae^{i(\omega\tau+\phi)} + Ae^{i(\omega\tau+\phi)} = (-\omega^2 A + 2\zeta i\omega A + A)e^{i(\omega\tau+\phi)} = e^{i\omega\tau}.$$

$$-\omega^2 A + 2\zeta i\omega A + A = e^{-i\phi} = \cos\phi - i\sin\phi.$$

Steady State Continued

$$A(1-\omega^2) = \cos\phi \qquad 2\zeta\omega A = -\sin\phi.$$

$$\left.\begin{array}{c} A^2(1-\omega^2)^2 = \cos^2\phi \\ (2\zeta\omega A)^2 = \sin^2\phi \end{array}\right\} \Rightarrow A^2[(1-\omega^2)^2 + (2\zeta\omega)^2] = 1.$$

$$A = A(\zeta,\omega) = \frac{1}{\sqrt{(1-\omega^2)^2 + (2\zeta\omega)^2}}.$$

Solve for Phase

$$\tan\phi = -\frac{2\zeta\omega}{1-\omega^2} = \frac{2\zeta\omega}{\omega^2-1} \Rightarrow \phi \equiv \phi(\zeta,\omega) = \arctan\left(\frac{2\zeta\omega}{\omega^2-1}\right).$$

Note the phase shift is frequency dependent
At low freq ϕ -> 0
At high freq ϕ -> 180 degrees
Remember $\omega = \omega/\omega_0$

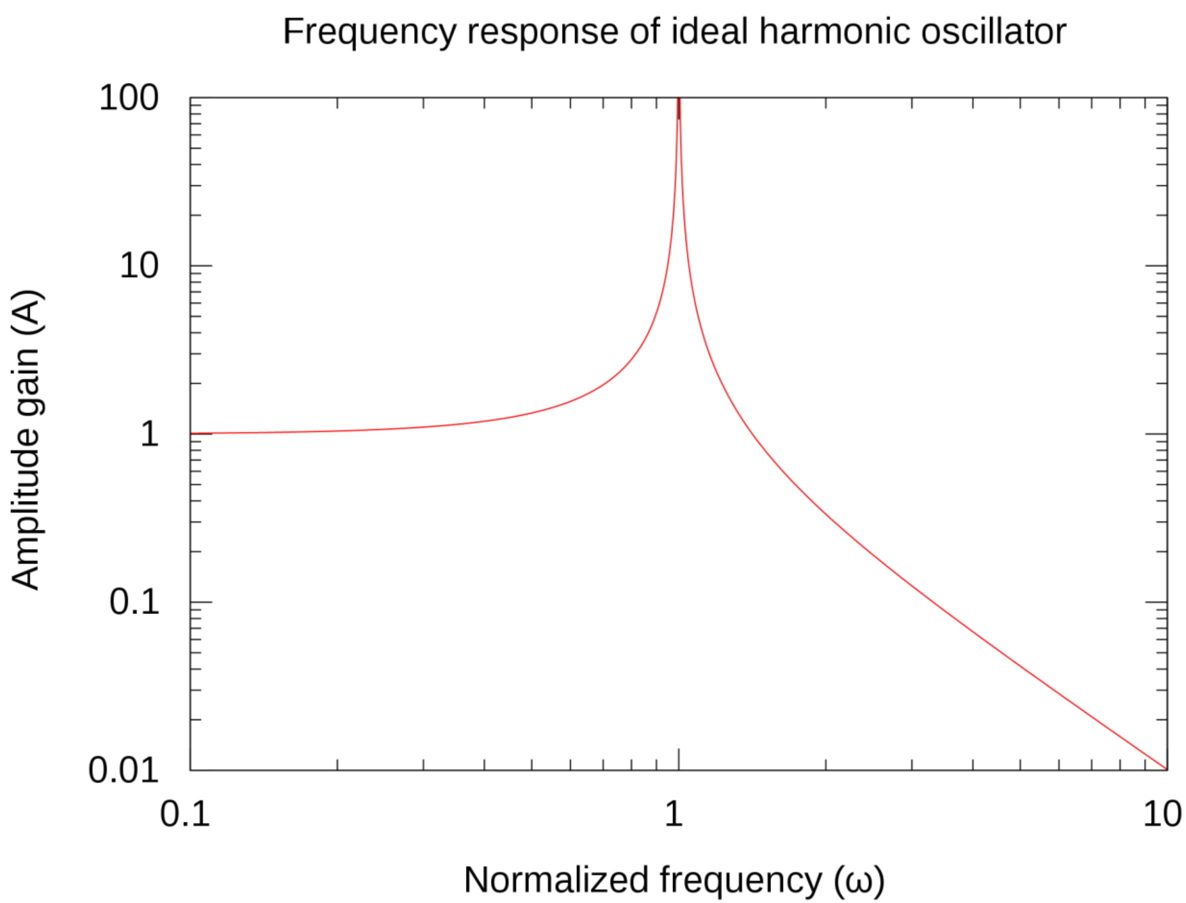
Full solution

$$q_s(\tau) = A(\zeta, \omega)\cos(\omega\tau + \phi(\zeta, \omega)) = A\cos(\omega\tau + \phi).$$

$$q(\tau) = q_t(\tau) + q_s(\tau).$$

Amplitude vs freq – Bode Plot

$$A = A(\zeta, \omega) = \frac{1}{\sqrt{(1-\omega^2)^2 + (2\zeta\omega)^2}}.$$



Various Damped Osc Systems

Translational Mechanical	Torsional Mechanical	Series RLC Circuit	Parallel RLC Circuit
Position x	Angle θ	Charge q	Voltage e
Velocity dx/dt	Angular velocity $d\theta/dt$	Current dq/dt	de/dt
Mass m	Moment of inertia I	Inductance L	Capacitance C
Spring constant K	Torsion constant μ	Elastance 1/C	Susceptance 1/L
Friction γ	Rotational friction Γ	Resistance R	Conductance 1/R
Drive force F(t)	Drive torque τ(t)	e(t)	di/dt

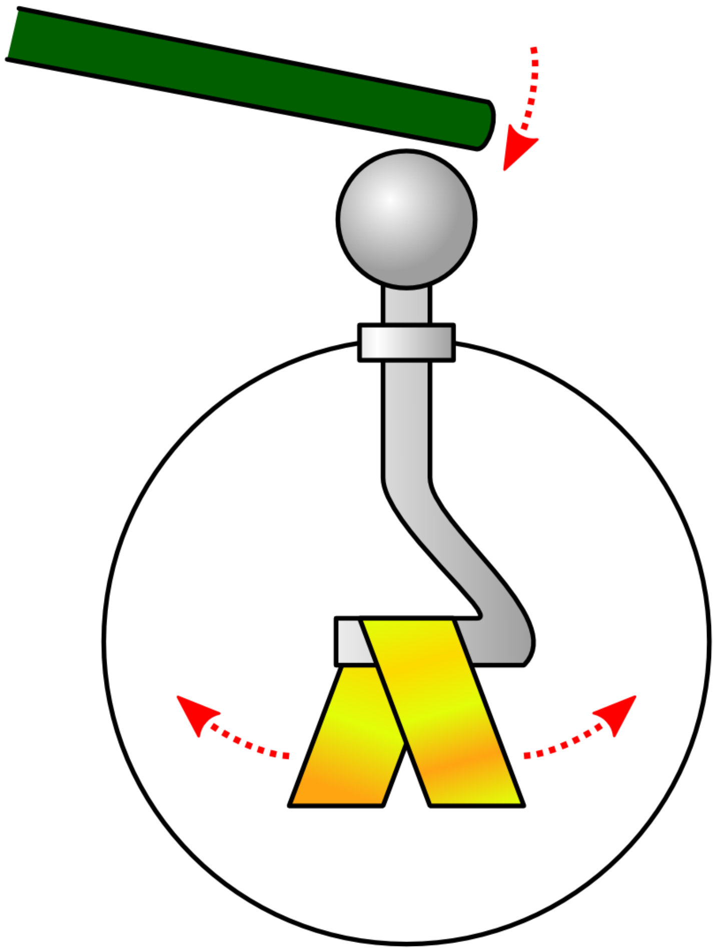
Undamped resonant frequency : f_n

$\frac{1}{2\pi}\sqrt{\frac{K}{M}}$	$\frac{1}{2\pi}\sqrt{\frac{\mu}{I}}$	$\frac{1}{2\pi}\sqrt{\frac{1}{LC}}$	$\frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

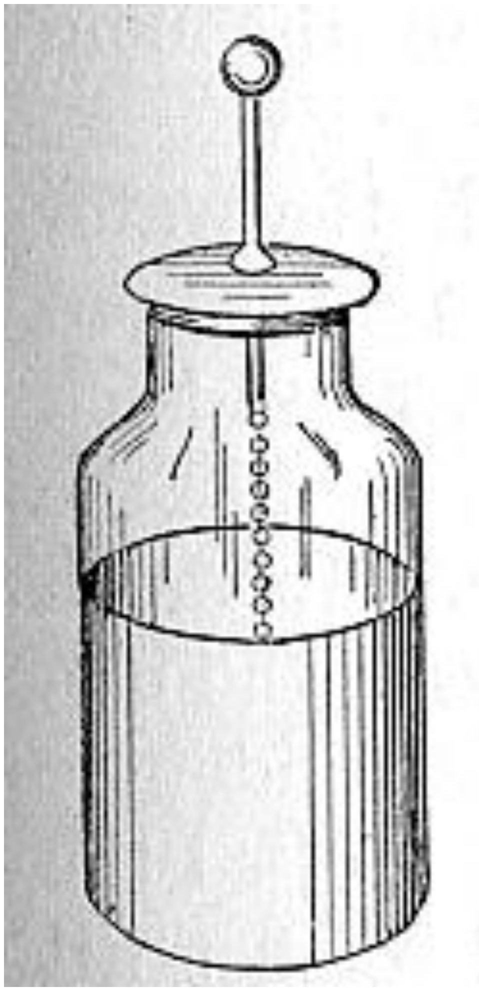
Differential equation:

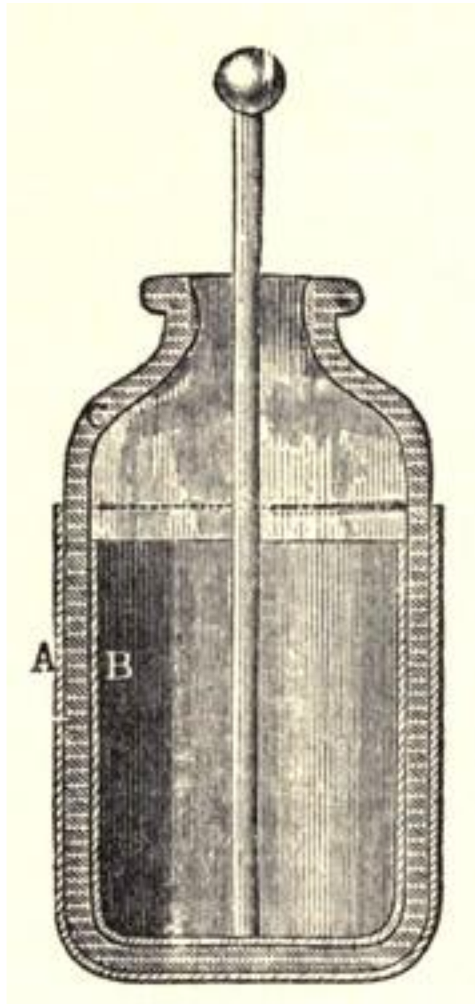
$M\ddot{x}+\gamma\dot{x}+Kx=F$	$I\ddot{\theta}+\Gamma\dot{\theta}+\mu\theta=\tau$	$L\ddot{q}+R\dot{q}+q/C=e$	$C\ddot{e}+\dot{e}/R+e/L=\dot{i}$

Gold leaf electroscope – used to show presence of charge
Gold leaf for gilding is about 100 nm thick!!



Leyden Jar – historical capacitor





Force between charges as measured on the lab with a torsion balance

$$F_C = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

ε_0 ~ 8.854 187 817 … x 10^{-12} Vacuum permittivity

$$\varepsilon_0 = \frac{1}{\mu_0 c_0^2}$$

μ_0 = Vacuum permeability (magnetic)
=4π 10^{-7} H m^{-1} – defined exactly
c_0 = speed of light in vacuum

Coulombs “Law”

Define the electric field E = F/q where F is the force on a charge q
In the lab we measure an inverse square force law like gravity

For a point charge Q the E field at a distance r is given by Coulomb’s Law. It is a radial field and points away from a positive charge and inward towards a negative charge

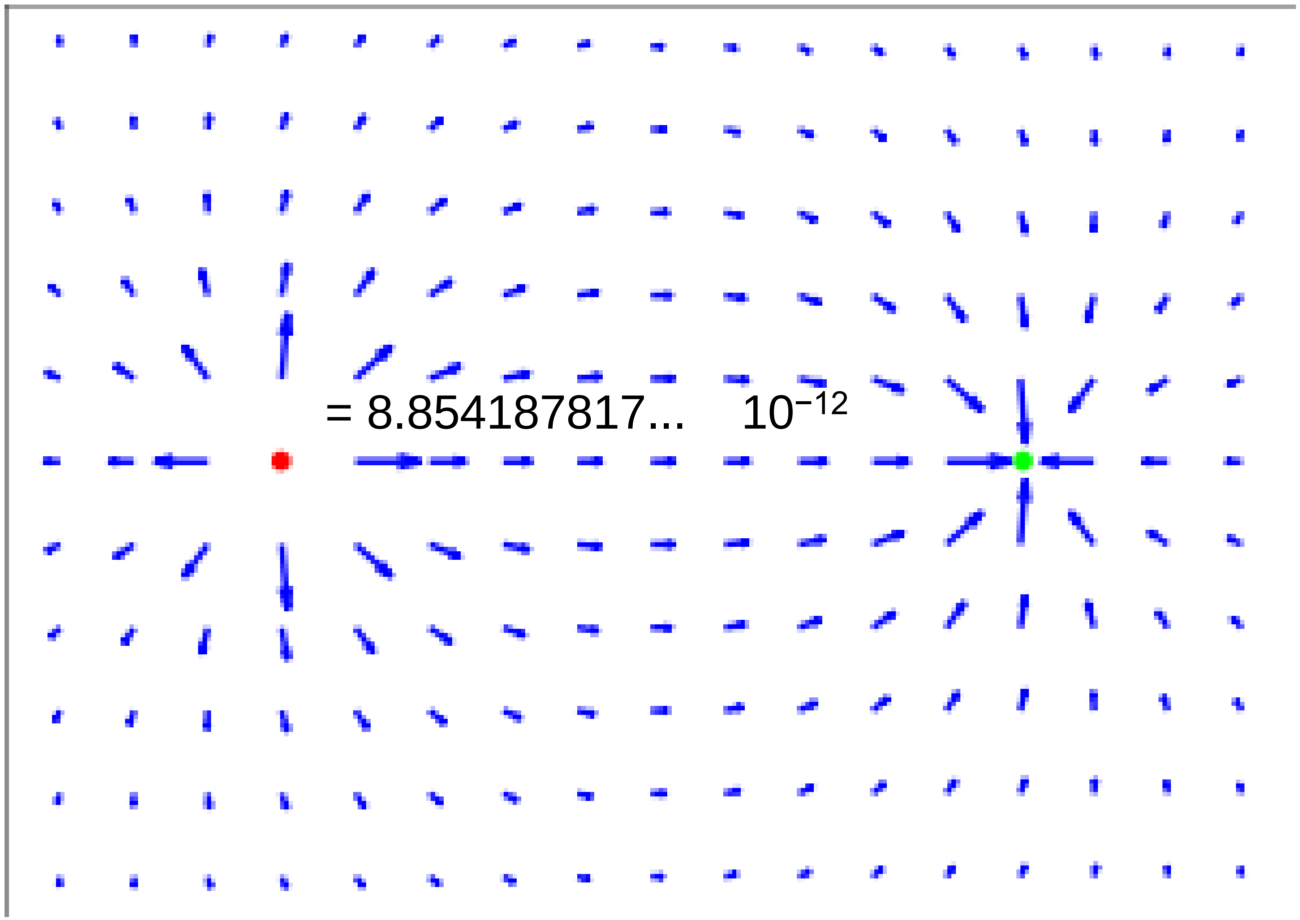
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \qquad (1)$$

Similarity to Newtons “Law” of Gravity
Both Coulomb and Newton are inverse square laws

$$\mathbf{F} = G\frac{Mm}{r^2}\hat{\mathbf{r}} = m\mathbf{g}$$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}} = q\mathbf{E}$$

Two charges – a dipole



Energy density in the electric field

$$u = \frac{1}{2}\varepsilon|\mathbf{E}|^2 \, ,$$

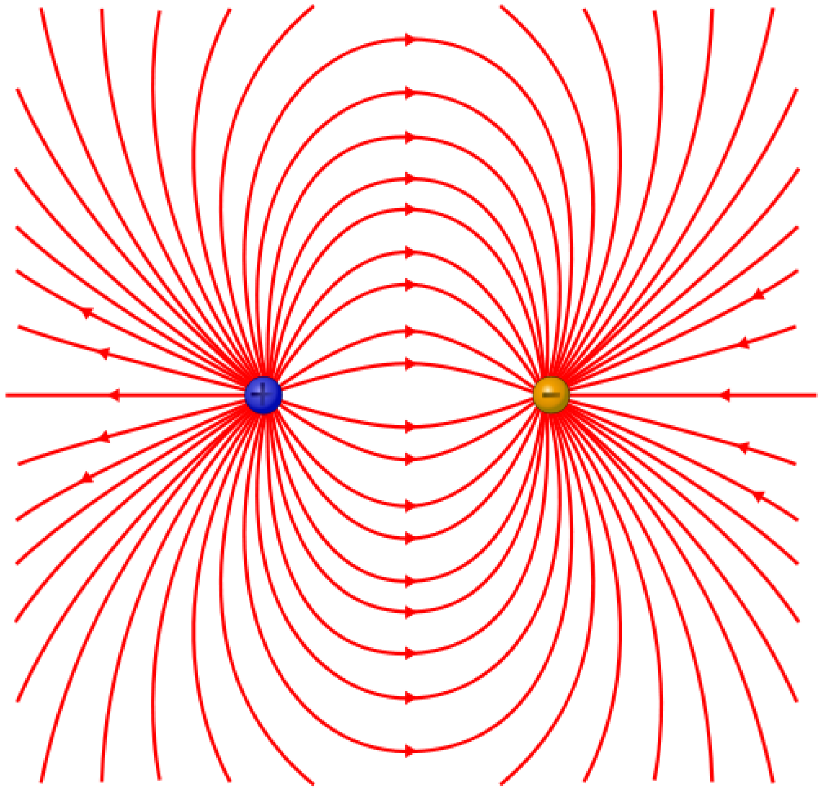
Energy per unit volume J/m^3

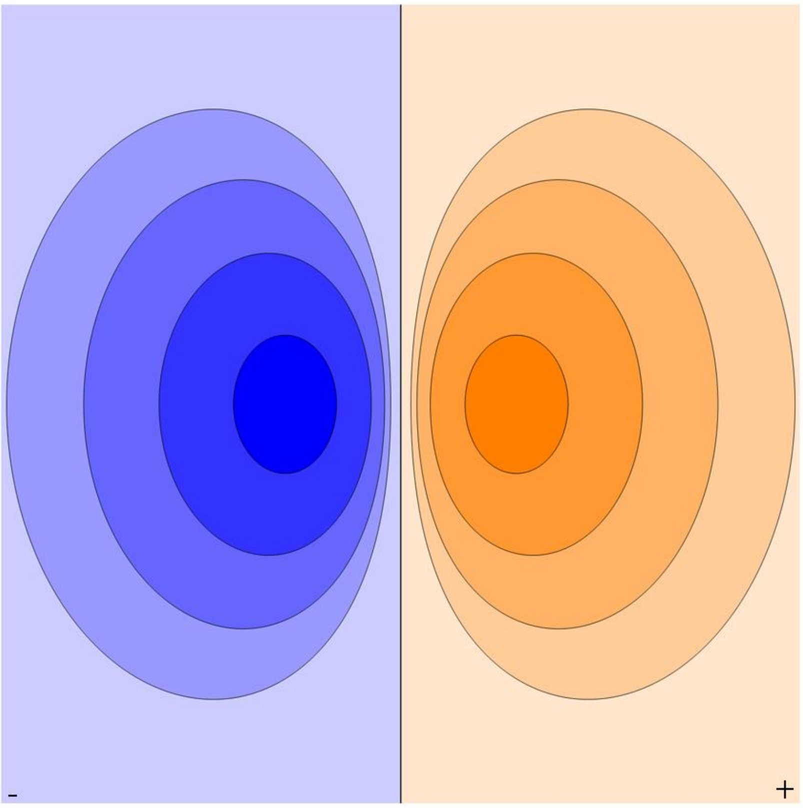
$$\frac{1}{2}\varepsilon\int_V |\mathbf{E}|^2 \, \mathrm{d}V \, ,$$

Total energy in a volume - Joules

Dipoles

Electric Field Lines - Equipotentials





Dipole moment definition

We define the dipole moment **p** (vector) for a set of charges q_i at vector positions $\mathbf{r}_i$ as:

$$\mathbf{p} = \sum_{i=1}^{N} q_i \, \mathbf{r}_i \, .$$

For two equal and opposite charges (q) we have p=q*r where r is the distance between them

Multipole Expansions

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_l^m Y_l^m(\theta, \phi).$$

General spherical harmonic expansion of a function on the unit sphere Y_{lm} functions are called spherical harmonics – like a Fourier transform but done on a sphere not a flat surface. C_{lm} are coefficients

l=0 is a monopole
l=1 is a dipole
l=2 is a quadrupole
l=3 is an octopole (also spelled octupole)