Homework 1: Phys 141

April 15, 2019

Problem 1.2
Starting from the wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2}$$

if $f = f(z - ut)$, then we can use the chain rule to show that

$$\frac{\partial f}{\partial z} = f'(z-ut)\frac{\partial(z-ut)}{\partial z} = f'(z-ut)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial f'(z-ut)}{\partial z} = f''(z-ut)\frac{\partial(z-ut)}{\partial z} = f''(z-ut)$$

Similarly,

$$\frac{\partial f}{\partial t} = f'(z-ut)\frac{\partial(z-ut)}{\partial t} = -uf'(z-ut)$$

$$\frac{\partial^2 f}{\partial t^2} = -u\frac{\partial f'(z-ut)}{\partial t} = -uf''(z-ut)\frac{\partial(z-ut)}{\partial t} = u^2 f''(z-ut)$$

These clearly satisfy the wave equation.
Problem 1.3
In three dimensions, we have

$$\frac{1}{u^2}\frac{\partial^2 f(\hat{n}\cdot\overrightarrow{r} - ut)}{\partial t^2} = \nabla^2 f(\hat{n}\cdot\overrightarrow{r} - ut) = \nabla^2 f(n_x x + n_y y + n_z z - ut) =$$

$$\frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial x^2} + \frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial y^2} + \frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial z^2}$$

Now, all of these derivatives can be done as in the previous problem,

$$\frac{\partial^2 f(\hat{n}\cdot\overrightarrow{r} - ut)}{\partial t^2} = u^2 f''$$

$$\frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial x^2} = n_x^2 f''$$

$$\frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial y^2} = n_y^2 f''$$

$$\frac{\partial^2 f(n_x x + n_y y + n_z z - ut)}{\partial z^2} = n_z^2 f''$$

Putting these back into the wave equation, we find

$$\frac{1}{u^2} u^2 f'' = (n_x^2 + n_y^2 + n_z^2) f''$$

Which clearly holds, since $\hat{n}$ is a unit vector.

Problem 1.4

Starting from the 3D wave equation,

$$\nabla^2 f = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2}$$

In spherical coordinates,

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial f}{\partial r}\right] + \text{Derivatives in } \theta \text{ and } \phi$$

Since our function, $f = \frac{1}{r} e^{i(kr-\omega t)}$ does not depend on θ or ϕ, the terms containing derivatives with respect to θ and ϕ will be zero. This leaves

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\frac{1}{r} e^{i(kr-\omega t)}\right] = \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{r} e^{i(kr-\omega t)}\right]$$

The right hand side becomes

$$\frac{1}{u^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{r} e^{i(kr-\omega t)}\right] = -\frac{1}{r}\frac{\omega^2}{u^2} e^{i(kr-\omega t)}$$

The left hand side becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\frac{1}{r} e^{i(kr-\omega t)}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\Big(ikr - 1\Big) e^{i(kr-\omega t)}\right] = -\frac{1}{r} k^2 e^{i(kr-\omega t)}$$

Putting these together,

$$k^2 e^{i(kr-\omega t)} = \frac{\omega^2}{u^2} e^{i(kr-\omega t)}$$

Which holds if $u = \frac{\omega}{k}$.

Problem 1.6

The book leaves the derivation at

$$u_g = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{dk} = u - \frac{ck}{n^2}\frac{dn}{dk}$$

By the chain rule

$$u_g = u - \frac{ck}{n^2}\frac{dn}{du}\frac{d\lambda}{dk}\frac{du}{d\lambda}$$

Using that:

$$n = \frac{c}{u} \Rightarrow \frac{dn}{du} = -\frac{c}{u^2}$$

and

$$\lambda = \frac{2\pi}{k} \Rightarrow \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

we find

$$u_g = u - \frac{ck}{n^2}\frac{c}{u^2}\frac{2\pi}{k^2}\frac{du}{d\lambda} = u - \frac{2\pi}{k}\frac{du}{d\lambda} = u - \lambda\frac{du}{d\lambda}$$

To derive the second identity, we start from

$$\frac{1}{u_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u}\right)$$

$$= \frac{1}{u} + \omega\frac{d}{d\omega}\left(\frac{1}{u}\right) = \frac{1}{u} + \omega\frac{d}{du}\left(\frac{1}{u}\right)\frac{du}{d\omega} = \frac{1}{u} - \frac{\omega}{u^2}\frac{du}{d\omega}$$

Using the chain rule,

$$\frac{1}{u} - \frac{\omega}{u^2}\frac{du}{d\omega} = \frac{1}{u} - \frac{\omega}{u^2}\frac{du}{dn}\frac{d\lambda_0}{d\omega}\frac{dn}{d\lambda_0}$$

Using that

$$u = \frac{c}{n} \Rightarrow \frac{du}{dn} = -\frac{c}{n^2}$$

and

$$\lambda_0 = \frac{2\pi}{k} = \frac{2\pi u}{\omega} \Rightarrow \frac{d\lambda_0}{d\omega} = -\frac{2\pi u}{\omega^2}$$

we find

$$\frac{1}{u} - \frac{\omega}{u^2}\frac{du}{dn}\frac{d\lambda_0}{d\omega}\frac{dn}{d\lambda_0} = \frac{1}{u} - \frac{\omega}{u^2}\frac{c}{n^2}\frac{2\pi u}{\omega^2}\frac{dn}{d\lambda_0} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}$$

Problem 1.7

We start from the equation

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}$$

In this case,

$$\frac{dn}{d\lambda_0} = -2B\lambda_0^{-3}$$

$$\Rightarrow \frac{1}{u_g} = \frac{n}{c} + \frac{2B\lambda_0^{-2}}{c} = \frac{A}{c} + \frac{B\lambda_0^{-2}}{c} + \frac{2B\lambda_0^{-2}}{c} = \frac{A}{c} + \frac{3B\lambda_0^{-2}}{c}$$

Putting in numerical values, we find that
$u_g = 1.6x10^8 \frac{m}{s}$
Problem 1.8
We start from:

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}$$

If $n = \frac{A}{\lambda_0}$, then $\frac{dn}{d\lambda_0} = -\frac{A}{\lambda_0^2}$. So,

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0} = \frac{1}{u} + \frac{\lambda_0}{c}\frac{A}{\lambda_0^2} = \frac{1}{u} + \frac{A}{c\lambda_0} = \frac{1}{u} + \frac{n}{c} = \frac{2}{u}$$

So,

$$u_g = \frac{u}{2}$$

Problem 2.1
The function $f = e^{i(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t)}$ can be expanded to

$$f = e^{i(k_x x + k_y y + k_z z - \omega t)}$$

Applying the del operator

$$\overrightarrow{\nabla} f = \frac{d}{dx}e^{i(k_x x + k_y y + k_z z - \omega t)}\hat{x} + \frac{d}{dy}e^{i(k_x x + k_y y + k_z z - \omega t)}\hat{y} + \frac{d}{dz}e^{i(k_x x + k_y y + k_z z - \omega t)}\hat{z}$$

$$= ik_x\hat{x}e^{i(k_x x + k_y y + k_z z - \omega t)} + ik_y\hat{y}e^{i(k_x x + k_y y + k_z z - \omega t)} + ik_z\hat{z}e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(k_x\hat{x} + k_y\hat{y} + k_z\hat{z})e^{i(k_x x + k_y y + k_z z - \omega t)} = i\hat{k}f$$

Problem 2.2
The magnitude of the average value of the Poynting vector is given by

$$\left|\left\langle\overrightarrow{S}\right\rangle\right| = \frac{1}{2}\left|\overrightarrow{E}X\overrightarrow{H}\right| = \frac{\epsilon_0 c}{2}\left|\overrightarrow{E}X\overrightarrow{E}\right| = \frac{\epsilon_0 c}{2}\left|\overrightarrow{E}^2\right| = I = \frac{P}{A}$$

So then, the RMS is

$$\sqrt{|\overrightarrow{E}^2|} = \sqrt{\frac{2P}{\epsilon_0 cA}} = 154.8\frac{V}{m}$$

Problem 2.3
Starting with the magnitude of the Poynting vector

$$I = \left|\left\langle\overrightarrow{S}\right\rangle\right| = \frac{\text{Power}}{\text{Laser Area}} = \frac{100 * 10^6}{\pi(5 * 10^{-6})^2}\frac{W}{m^2} = 1.3 * 10^{18}\frac{W}{m^2}$$

In the previous problem, we showed that

$$I = \frac{\epsilon_0 c}{2}\left|\overrightarrow{E}^2\right|$$

$$\Rightarrow \sqrt{|\overrightarrow{E}^2|} = \sqrt{\frac{2I}{\epsilon_0 c}} = 3.1 * 10^{10}\frac{V}{m}$$

Problem 2.4

We start with the definition of the Poynting vector.

$$\overrightarrow{S} = \overrightarrow{E} X \overrightarrow{H}$$

$$= Re\Big[E_0 e^{i(k\cdot r-\omega t)}\Big] X Re\Big[H_0 e^{i(k\cdot r-\omega t)}\Big]$$

$$= \frac{1}{2}\Big[E_0 e^{i(k\cdot r-\omega t)} + E_0^* e^{-i(k\cdot r-\omega t)}\Big] X \Big[H_0 e^{i(k\cdot r-\omega t)} + H_0^* e^{-i(k\cdot r-\omega t)}\Big]$$

$$= \frac{1}{4}\Big[E_0 X H_0 e^{2i(k\cdot r-\omega t)} + E_0^* X H_0^* e^{-2i(k\cdot r-\omega t)} + E_0^* X H_0 + E_0 X H_0^*\Big]$$

$$= \frac{1}{4}\Big[E_0^* X H_0 + E_0 X H_0^*\Big] + \frac{1}{4}\Big[E_0 X H_0 e^{2i(k\cdot r-\omega t)} + E_0^* X H_0^* e^{-2i(k\cdot r-\omega t)}\Big]$$

$$= \frac{1}{2} Re(E_0 X H_0^*) + \frac{1}{2} Re(E_0 X H_0 e^{2i(k\cdot r-\omega t)})$$

So then,

$$\Big\langle \overrightarrow{S} \Big\rangle = \frac{1}{2}\Big\langle Re(E_0 X H_0^*)\Big\rangle + \frac{1}{2}\Big\langle Re(E_0 X H_0 e^{2i(k\cdot r-\omega t)})\Big\rangle$$

The average is taken over one cycle, $T = \frac{2\pi}{\omega}$. The second term oscillates are twice this frequency, meaning the average is zero. This leaves only the first term, giving

$$\Big\langle \overrightarrow{S} \Big\rangle = \frac{1}{2}\Big\langle Re(E_0 X H_0^*)\Big\rangle$$