

# Homework 4: Phys 141

June 6, 2019

## Problem 5.1

The criterion for Fresnel versus Fraunhofer diffraction is given by

$$\frac{1}{2} \left( \frac{1}{d'} + \frac{1}{d} \right) \delta^2 \ll \lambda$$

In this problem,  $\lambda = 600nm$ ,  $d = 10m$  and  $\delta = 1mm$ .  
in part a)

$$d' = 1cm$$

, giving

$$\frac{1}{2} \left( \frac{1}{d'} + \frac{1}{d} \right) \delta^2 = \frac{1}{2} \left( \frac{1}{10} + \frac{1}{0.01} \right) (0.001)^2 = 5 \times 10^{-5} m$$

Which is not much smaller than  $6 \times 10^{-7} m$ . So Fresnel diffraction applies.  
in part b)

$$d' = 2m$$

, giving

$$\frac{1}{2} \left( \frac{1}{d'} + \frac{1}{d} \right) \delta^2 = \frac{1}{2} \left( \frac{1}{10} + \frac{1}{2} \right) (0.001)^2 = 3 \times 10^{-7} m$$

Which is smaller than  $6 \times 10^{-7} m$ . So Fraunhofer diffraction applies.

## Problem 5.2

We start from  $\beta = \frac{1}{2} kb \sin \theta$ , where  $b = 0.5mm$  and  $k = \frac{2\pi}{\lambda} = 9.9 \times 10^6$ .  
Now,

$$\tan(\theta) = \frac{x}{d}$$

where  $x$  is the distance from the central maximum to some point on the screen.  
So,

$$x = d \tan(\theta) = d \tan \left( \arcsin \left( \frac{2\beta}{kb} \right) \right)$$

The first minimum occurs at  $\beta = \pi$ , so

$$x_1^{min} = d \tan(\theta) = d \tan\left(\arcsin\left(\frac{2\pi}{kb}\right)\right) = 0.633mm$$

The second maximum occurs at  $\beta = 1.5\pi$ , giving

$$x_1^{min} = d \tan(\theta) = d \tan\left(\arcsin\left(\frac{3\pi}{kb}\right)\right) = 0.905mm$$

**Problem 5.5**

The intensity on the screen is given by the equation

$$I(\beta) = I_0 \left(\frac{\sin(\beta)}{\beta}\right)^2$$

Setting the derivative with respect to  $\beta$  equal to zero,

$$\frac{dI(\beta)}{d\beta} = I_0 \frac{2\theta \sin(\beta) \cos(\beta) - 2 \sin^2(\beta)}{\beta^3} = 0$$

$$\Rightarrow 2 \sin(\beta)(\beta \cos(\beta) - \sin(\beta)) = 0$$

$$\Rightarrow \beta \cos(\beta) = \sin(\beta)$$

$$\Rightarrow \tan(\beta) = \beta$$

The easiest way to find the roots of this transcendental equation is using some root finding algorithm.

We can rewrite the above equation as

$$\beta = \arctan(\beta) = \arctan(\beta + n\pi)$$

For large  $\beta$ , we can expand to

$$\beta = \left(n + \frac{1}{2}\right)\pi + \dots$$

**Problem 5.7**

Say that the angular resolution of the starts is  $\theta$ . Then,

$$\tan\left(\frac{\theta}{2}\right) = \frac{d_{startostar}}{10ly}$$

$$\Rightarrow \theta = 2.11 \times 10^{-6}$$

Now,

$$\theta = \frac{1.22\lambda}{R}$$

$$R = \frac{1.22\lambda}{\theta} = 0.29m$$

**Problem 5.12**

For a double slit aperture

$$I(\beta, \gamma) = I_0 \left( \frac{\sin(\beta)}{\beta} \right)^2 \cos^2(\gamma)$$

Where  $\beta = \frac{1}{2}kb \sin(\theta)$  and  $\gamma = \frac{1}{2}kh \sin(\theta)$ . So,  $\beta = \frac{b}{h}\gamma$ .

For the central diffraction envelope,  $\beta = [-\pi, \pi]$ . This gives  $\gamma$  a range of  $\gamma = [-\frac{h}{b}\pi, \frac{h}{b}\pi]$ .

We are looking for values of  $\gamma$  where  $I(\beta, \gamma)$  is at a maximum with respect to  $\gamma$ , meaning where  $\cos^2(\gamma) = 1$ . Now,  $\cos^2(\gamma) = 1$  at  $\gamma = \pi n$ .

Over the range  $\gamma = [0, \frac{h}{b}\pi]$ ,  $\gamma = n\pi \frac{h}{b} + 1$  times.

By symmetry, we get a total of

$$n = 2 \left( \frac{h}{b} + 1 \right)$$

**Problem 5.17**

Starting from the equation

$$U_p = -\frac{ik}{4\pi} \int \frac{U_0 e^{ik(r+r')-i\omega t}}{rr'} [\cos(\mathbf{n}, \mathbf{r}) + 1] dA$$

Lets use the coordinates set up in Figure 5.20. First, we see that  $r = \sqrt{R^2 + h^2}$  and  $r' = \sqrt{R^2 + h'^2}$ . Writing our integral in cylindrical coordinates  $(\rho, \phi)$ , we get

$$U_p = -\frac{ik}{4\pi} e^{-i\omega t} U_0 \int_0^{2\pi} \int_0^R \frac{e^{ik(r+r')}}{\sqrt{\rho^2 + h^2} \sqrt{\rho^2 + h'^2}} [\cos(\mathbf{n}, \mathbf{r}) + 1] \rho d\rho d\phi$$

Now, we can write  $r + r' = h + h' + \frac{R^2}{2L}$

$$U_p = -\frac{ik}{4\pi} e^{ik(h+h')-i\omega t} U_0 \int_0^{2\pi} \int_0^R \frac{e^{\frac{ik\rho^2}{2R}}}{\sqrt{\rho^2 + h^2} \sqrt{\rho^2 + h'^2}} [\cos(\mathbf{n}, \mathbf{r}) + 1] \rho d\rho d\phi$$

By cylindrical symmetry of the problem (assuming we are working on axis), the integral over  $\phi$  is trivial

$$U_p = -\frac{ik}{2} e^{ik(h+h')-i\omega t} U_0 \int_0^R \frac{e^{\frac{ik\rho^2}{2R}}}{\sqrt{\rho^2 + h^2} \sqrt{\rho^2 + h'^2}} [\cos(\mathbf{n}, \mathbf{r}) + 1] \rho d\rho$$

using the substitution  $\xi = \rho^2$ , we find

$$U_p = -\frac{ik}{4} e^{ik(h+h')-i\omega t} U_0 \int_0^{R^2} \frac{e^{\frac{ik\xi}{2R}}}{\sqrt{\xi + h^2} \sqrt{\xi + h'^2}} [\cos(\mathbf{n}, \mathbf{r}) + 1] d\xi$$

$$U_p = -\frac{ik}{4} e^{ik(h+h')-i\omega t} U_0 \int_0^{R^2} \frac{e^{\frac{i\pi\xi}{\lambda R}}}{\sqrt{\xi + h^2} \sqrt{\xi + h'^2}} [\cos(\mathbf{n}, \mathbf{r}) + 1] d\xi$$

This integral is still difficult to do, but can be evaluated by considering phasors. As we integrate, the exponential term inside the integral causes the the phase of the integral to

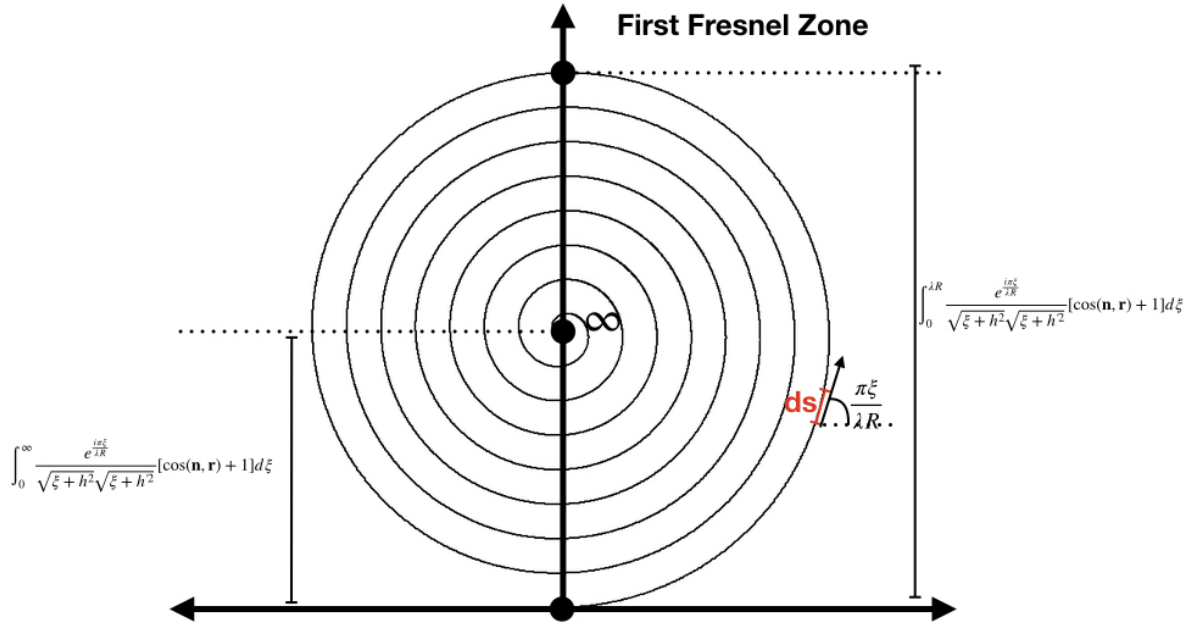


Figure 1:

change. The term inside the integral will change sign at each multiple of  $\xi = \lambda R$ . The other part of the integral  $\frac{\cos(\mathbf{n}, \mathbf{r}) + 1}{\sqrt{\xi + h^2} \sqrt{\xi + h'^2}}$  decreases as  $\xi$  increases.

This means that our integral is essentially a spiral in the complex plane. This is shown in figure 1.

If we integrate from 0 to the first Fresnel zone  $\xi = \lambda R$ , we remain on the outside of the spiral. If we integrate over all Fresnel zones, the integral continues to spiral inward. The integral over all Fresnel zones is half of the integral over the first Fresnel zone.

**Problem 5.19**

**Part a)**

Starting with

$$U_p = \frac{U_0}{(1+i)^2} \left[ C(u) + iS(u) \right] \Big|_{u_1}^{u_2} \left[ C(v) + iS(v) \right] \Big|_{v_1}^{v_2}$$

So, for a thin slit

$$U_p = \frac{U_0}{(1+i)^2} \left[ C(v) + iS(v) \right] \Big|_{v_1}^{v_2}$$

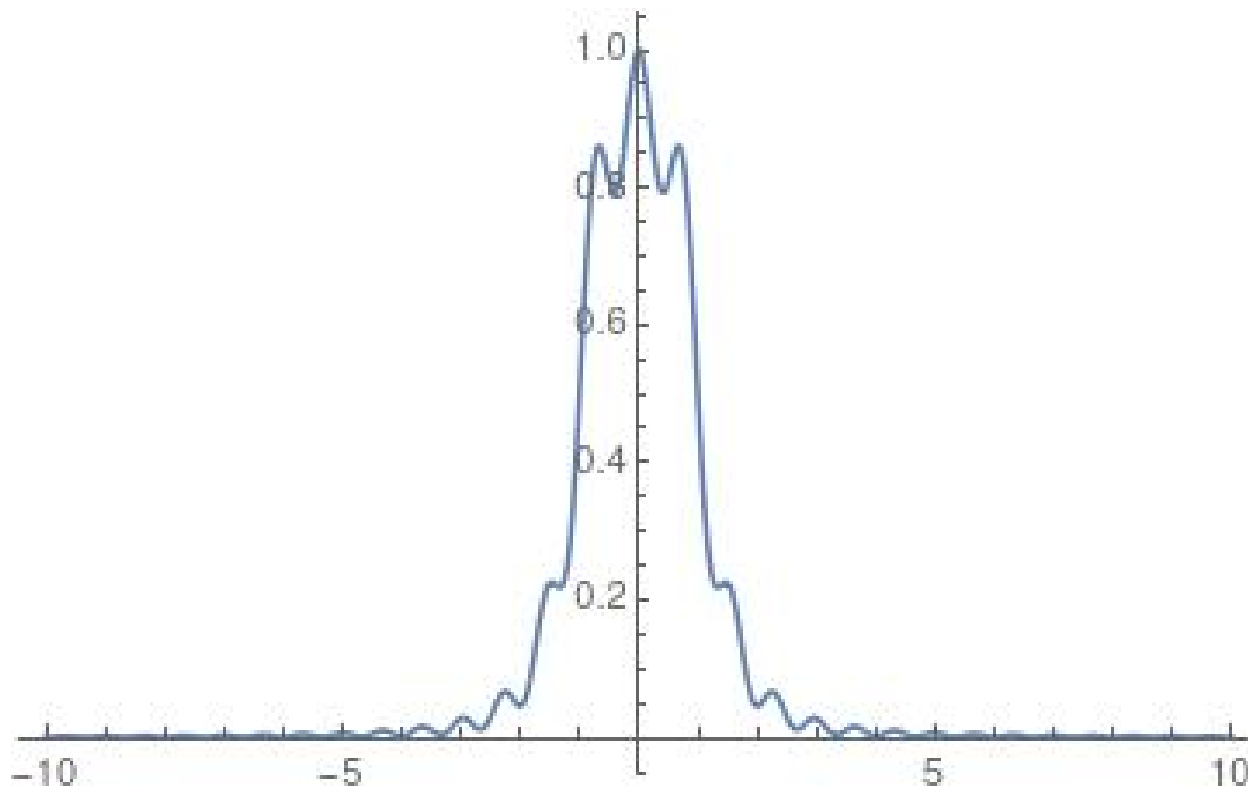


Figure 2:

Let's take the slit to be centered at zero, and  $\Delta v = 3$ , so that

$$U_p = \frac{U_0}{(1+i)^2} \left[ C(v) + iS(v) \right] \Big|_{x-1.5}^{x+1.5} = \frac{U_0}{(1+i)^2} \left[ C(x+1.5) - C(x-1.5) + iS(x+1.5) - iS(x-1.5) \right]$$

Now,  $I = |U_p|^2$ , giving

$$I = |U_p|^2 = \frac{U_0^2}{(1+i)^2(1-i)^2} \left[ C(x+1.5) - C(x-1.5) + iS(x+1.5) - iS(x-1.5) \right]^* \left[ C(x+1.5) - C(x-1.5) - iS(x+1.5) + iS(x-1.5) \right]$$

$$I = \frac{U_0^2}{4} \left[ (C(x+1.5) - C(x-1.5))^2 + (S(x+1.5) - S(x-1.5))^2 \right]$$

Putting this function into Mathematica, we get the following plot

**Part b)**

From Babinet's principle, we know that a complimentary aperture will produce an identical diffraction pattern. So the diffraction pattern will be the same.

**Part c)**

We start from the equation

$$v_2 - v_1 = (y_2 - y_1) \sqrt{\frac{2}{\lambda L}}$$
$$\Rightarrow L = \frac{2(y_2 - y_1)^2}{\lambda(v_2 - v_1)^2} = \left(\frac{1}{h} + \frac{1}{h'}\right)^{-1}$$

If the light rays are coming from infinity, then  $h' \rightarrow \infty$ , and  $L \rightarrow h$

$$h = \frac{2(y_2 - y_1)^2}{\lambda(v_2 - v_1)^2} = \frac{2(1 * 10^{-3})^2}{(500 * 10^{-9})(3)^2} = \frac{4}{9}m$$