Homework 5: Phys 141

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Problem 6.1
We begin from the equations

$$n^2 - \kappa^2 = 1 + \frac{Ne^2}{m\epsilon_0}\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

$$2n\kappa = \frac{Ne^2}{m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

In the case that $\kappa \ll n$, we must be far away from resonance, so that $(\omega_0^2 - \omega^2)^2 \gg \gamma^2\omega^2$.

$$n^2 - \kappa^2 \approx n^2 = 1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2}$$

$$\Rightarrow n = \sqrt{1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2}}$$

And since $\omega_0^2 - \omega^2 \gg \frac{Ne^2}{m\epsilon_0}$ we Taylor expand to find

$$n \approx 1 + \frac{Ne^2}{2m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2}$$

So then,

$$n\kappa \approx \kappa\left(1 + \frac{Ne^2}{2m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right) = \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2} \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2}$$

Solving for κ gives

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)(\frac{Ne^2}{2m\epsilon_0} + \omega_0^2 - \omega^2)}$$

Again, since $\omega_0^2 - \omega^2 \gg \frac{Ne^2}{m\epsilon_0}$, we find that

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2}$$

Problem 6.3

Starting from the equation

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \left(\frac{f_j}{\omega_j^2 - \omega^2}\right)$$

Writing this equation in terms of wavelength, we find

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \left(\frac{f_j}{(\frac{2\pi c}{\lambda_j})^2 - (\frac{2\pi c}{\lambda})^2}\right) = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \left(\frac{f_j\lambda_j^2\lambda^2}{(2\pi c\lambda)^2 - (2\pi c\lambda_j)^2}\right)$$

$$\Rightarrow n^2 = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \left(\frac{\frac{f_j\lambda_j^2}{4\pi^2c^2}\lambda^2}{\lambda^2 - \lambda_j^2}\right) = 1 + \sum_j \left(\frac{A_j\lambda^2}{\lambda^2 - \lambda_j^2}\right)$$

Problem 6.6
Part a)
The plasma frequency is given by the equation

$$\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_0}} = \sqrt{\frac{(1.5*10^{28})e^2}{m\epsilon_0}} = 6.91*10^{15}\frac{m}{s}$$

Part b)
The relaxation time is given by

$$\tau = \frac{\sigma m}{Ne^2} = \frac{6.8*10^7 m_e}{1.5*10^{28}q^2} = 1.61*10^{-13}s$$

Part c)
The real and imaginary parts of the index of refraction can be written as

$$n^2 - \kappa^2 = 1 - \frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

$$2n\kappa = \frac{\omega_p^2}{\omega^2 + \tau^{-2}}\left(\frac{1}{\omega\tau}\right)$$

At $\lambda = 1\mu m$, $\omega = 1.88*10^{15}$.
We can then solve to find $n = 0.006, \kappa = 3.5$.
Part d)
The normal reflectance is given by the formula

$$R = \frac{(1-n)^2 + \kappa^2}{(1+n)^2 + \kappa^2} = \frac{(0.994)^2 + 3.5^2}{(1.006)^2 + 3.5^2} = 0.998$$

Problem 6.7
On normal incidence

$$r = \frac{1 - n - i\kappa}{1 + n + i\kappa} = \frac{(1 - n - i\kappa)(1 + n - i\kappa)}{(1+n)^2 + \kappa^2} = \frac{1 - n^2 - \kappa^2 - 2i\kappa}{(1+n)^2 + \kappa^2} = |r|e^{-i\alpha}$$

Where α is the phase shift upon reflection. Rewriting the exponential

$$\frac{1 - n^2 - \kappa^2 - 2i\kappa}{(1+n)^2 + \kappa^2} = |r|(\cos(\alpha) + i\sin(\alpha))$$

So,

$$|r|\cos(\alpha) = \frac{1 - n^2 - \kappa^2}{(1+n)^2 + \kappa^2}$$

and

$$|r|\sin(\alpha) = \frac{-2\kappa}{(1+n)^2 + \kappa^2}$$

So,

$$\tan(\alpha) = \frac{2\kappa}{n^2 + \kappa^2 - 1}$$

Problem 6.8

The normal reflectance is given by the equation

$$R = \frac{(1-n)^2 + \kappa^2}{(1+n)^2 + \kappa^2} = \frac{(0.5)^2 + 3.2^2}{(2.5)^2 + 3.2^2} = 0.64$$

The phase change on reflection is given by the equation

$$\arctan\left(\frac{2\kappa}{n^2 + \kappa^2 - 1}\right) = \arctan\left(\frac{2 * 3.2}{1.5^2 + 3.2^2 - 1}\right) = 0.508$$

Problem 10.1

We will use the figure below.

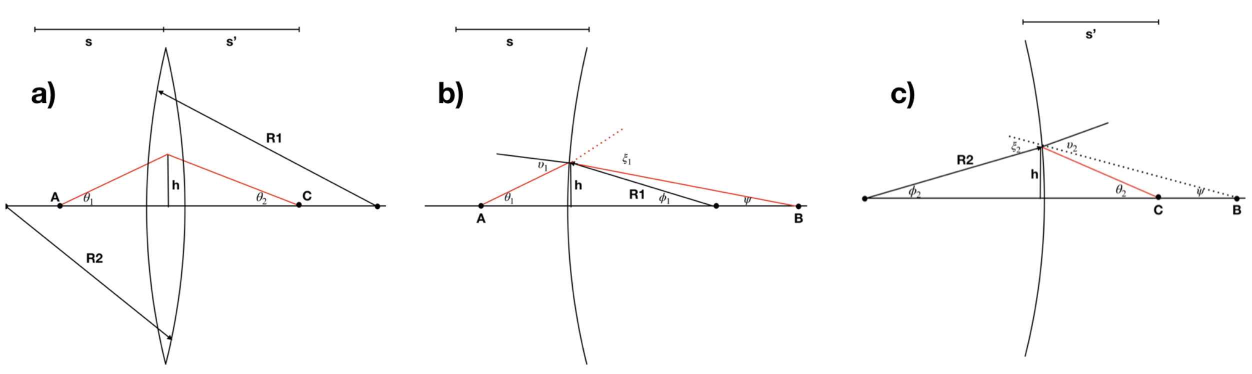

Figure 1:

We are imagining the light rays starting at point A. The rays are refracted off of the first surface of the lens, and are directed toward point B. They are then refracted off of the second second surface, toward point C.

Looking at the second picture, we start from Snell's law

$$\sin(\upsilon_1) = n\sin(\xi_1)$$

Using the small angle approximation, we find

$$\upsilon_1 = n\xi_1$$

Also, by geometry,

$$1)\upsilon_1 = \theta_1 + \phi_1 = n\xi_1$$

and

$$2)\xi_1 = \phi_1 - \psi$$

Combining 1 and 2, we find that

$$3)\theta_1 + \phi_1 = n\Big(\phi_1 - \psi\Big)$$

Looking at the third picture, we also use Snell's law

$$\sin(\upsilon_2) = n\sin(\xi_2)$$

Again, using the small angle approximation, we find

$$\upsilon_2 = n\xi_2$$

Also, by geometry,

$$4)\upsilon_2 = \theta_2 + \phi_2 = n\xi_2$$

and

$$5)\xi_2 = \phi_2 + \psi$$

We can combine 4 and 5 to arrive at the equation

$$6)\theta_2 + \phi_2 = n\Big(\phi_2 + \psi\Big)$$

Combining 3 and 6, we find that

$$\theta_1 + \theta_2 = (n-1)(\phi_1 + \phi_2)$$

Now, we can make the following approximations

$$\tan(\theta_1) \approx \theta_1 = \frac{h}{s}$$

$$\sin(\phi_1) \approx \phi_1 = \frac{h}{R_1}$$

$$\tan(\theta_2) \approx \theta_2 = \frac{h}{s'}$$

$$\sin(\phi_2) \approx \phi_2 = \frac{h}{R_2}$$

Which gives

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} = (n-1)\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)$$

Problem 10.2

For a thick lens, we have that

$$\frac{1}{f} = (n-1)\Big(\frac{1}{r_1} + \frac{1}{r_2} - \frac{(n-1)^2 t}{n r_1 r_2}\Big)$$

In the case of a lens with $r_1 = r_2 = r$, we find that

$$\frac{1}{f} = (n-1)\Big(\frac{2nr - (n-1)^2 t}{nr^2}\Big)$$

So then,

$$f = \Big(\frac{nr^2}{(n-1)(2nr - (n-1)^2 t)}\Big)$$

The focal planes $d_1 = d_2$, since $r_1 = r_2$, and are at the location

$$d = ft\frac{1-n}{r} = \Big(\frac{nr^2}{(n-1)(2nr-(n-1)^2 t)}\Big)t\frac{1-n}{r} = \Big(\frac{ntr}{2nr - (n-1)^2 t}\Big)$$

Problem 10.3

We are looking for a system of lenses which are achromatic over some range. This means that

$$\frac{\partial}{\partial\lambda}\frac{1}{f} = 0 = \frac{\partial}{\partial\lambda}\frac{1}{f_1} + \frac{\partial}{\partial\lambda}\frac{1}{f_2} = \frac{\partial n_1}{\partial\lambda}\Big(\frac{1}{r_{1,1}} - \frac{1}{r_{2,1}}\Big) + \frac{\partial n_2}{\partial\lambda}\Big(\frac{1}{r_{1,2}} - \frac{1}{r_{2,2}}\Big)$$

Using that $\Big(\frac{1}{r_{1,1}} - \frac{1}{r_{2,1}}\Big) = \frac{1}{f_1(n_1-1)}$ and $\Big(\frac{1}{r_{1,2}} - \frac{1}{r_{2,2}}\Big) = \frac{1}{f_2(n_2-1)}$, we can show

$$\frac{1}{f_1(n_1-1)}\frac{\partial n_1}{\partial\lambda} + \frac{1}{f_2(n_2-1)}\frac{\partial n_2}{\partial\lambda} = \frac{1}{f_1}\delta_1 + \frac{1}{f_2}\delta_2 = 0$$

Also, using the equation

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

we can solve the two systems to find that

$$f_1 = f\Big(1 - \frac{\delta_1}{\delta_2}\Big)$$

$$f_1 = f\left(1 - \frac{\delta_2}{\delta_1}\right)$$